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Monotonicity of the Period Function in Planar Separable Hamiltonian Systems: An Extension of Chicone's Criterion

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Abstract
We investigate the monotonicity of the period function associated with planar Hamiltonian systems of the form $H(x, y) = F(x) + G(y)$, where G is an even function. We derive explicit sufficient conditions, expressed solely in terms of the functions F, G, and their derivatives, ensuring the monotonicity of the period function associated with a nondegenerate center. Our results extend Chicone's classical criterion, originally established for potential systems, to a broader class of separable Hamiltonian systems. The effectiveness of the proposed criterion is illustrated through applications including polynomial, trigonometric, hyperbolic and relativistic Hamiltonian systems.

Keywords Period function · Monotonicity · Planar Hamiltonian systems · Separable Hamiltonians · Chicone's criterion

1 Introduction and Main Results

The study of the monotonicity of the period function is a classical problem in the qualitative theory of planar Hamiltonian systems, with important connections to the analysis of centers, bifurcations, and oscillatory behavior.

In this work, we consider planar Hamiltonian systems with a Hamiltonian function of the form

$$H(x, y) = F(x) + G(y), \tag{1}$$

where F and G are smooth functions, and G is even, that is, $G(-y) = G(y)$. Assume that $F(0) = G(0) = F'(0) = G'(0) = 0$, and $F''(0) > 0$, $G''(0) > 0$. Under these conditions, the associated Hamiltonian system

$$\begin{cases} x' = G'(y), \\ y' = -F'(x), \end{cases} \tag{2}$$

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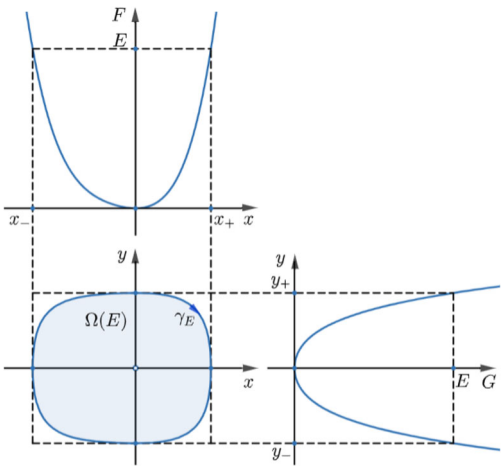


Fig. 1 Periodic orbit γ_E of energy E and the region $\Omega(E)$

has a nondegenerate center at $(0, 0)$, and its trajectories lie on the level curves $H(x, y) = E$. Let $\mathcal{P}$ denote the period annulus of the center, that is, the largest neighborhood of $(0, 0)$ filled with periodic orbits. It is well known (see, for instance, [1]) that $H(\mathcal{P}) = [0, E^*)$, where $E^* \in (0, +\infty]$ denotes the maximal energy level of the period annulus. If $E^* < +\infty$, then the boundary of $\mathcal{P}$ is contained in the energy level $H(x, y) = E^*$. Moreover, the periodic orbits in $\mathcal{P}$ can be parametrized by their energy, which allows us to define the period function $T : (0, E^*) \longrightarrow (0, +\infty)$, assigning to each periodic orbit its minimal period $T(E)$. The function T is smooth and satisfies $T(0) > 0$.

Let $0 < E_0 \leq E^*$. We say that T is *increasing* (respectively *decreasing*) on $(0, E_0)$ if $T'(E) \geq 0$ (respectively $T'(E) \leq 0$) for all $E \in (0, E_0)$. If $T'(E) = 0$ on $(0, E^*)$, then T is constant and the center is said to be *isochronous*.

Our main goal is to establish explicit conditions ensuring the monotonicity of the period function in terms of the functions F and G, extending Chicone's classical criterion to this broader Hamiltonian setting.

Let $\gamma_E \subset \mathcal{P}$ be the periodic orbit corresponding to the energy level E. This orbit intersects the axis $y = 0$ (respectively, $x = 0$) at the points determined by $F(x) = E$ (respectively, $G(y) = E$). Since F has a minimum at $x = 0$, for sufficiently small $E > 0$ the equation $F(x) = E$ has exactly two solutions: one positive, denoted by $x_+ = x_+(E)$, and one negative, denoted by $x_- = x_-(E)$. Analogously, we define $y_+ = y_+(E)$ and $y_- = y_-(E)$ (see Fig. 1).

For each $0 < E \leq E^*$, let

$$\Omega(E) = \{(x, y) \in \mathbb{R}^2 : 0 \leq H(x, y) \leq E\},$$

and define

$$\begin{aligned} M(x, y) = &\left[6F(x)(F''(x))^2 - 3(F'(x))^2F''(x) - 2F(x)F'(x)F'''(x)\right] G(y)\,(G'(y))^2 \\ &+ F(x)\,(F'(x))^2F''(x)\left[(G'(y))^2 - 2G(y)G''(y)\right]. \end{aligned}$$

The following theorem provides a sufficient condition for the period function to be monotone on $(0, E_0)$, where $0 < E_0 \leq E^*$, in terms of the sign of M on $\Omega(E_0)$.

Theorem 1.1 *Consider the planar separable Hamiltonian system given by*

$$H(x, y) = F(x) + G(y),$$

where G is even. Assume that the origin is a nondegenerate center, and let $T(E)$ be its period function.

Let $0 < E_0 \leq E^*$. If $M(x, y) \geq 0$ for all $(x, y) \in \Omega(E_0)$, then T is increasing on $(0, E_0)$. If $M(x, y) \leq 0$ for all $(x, y) \in \Omega(E_0)$, then T is decreasing on $(0, E_0)$.

In practice, one seeks the largest energy level $E_0 \leq E^*$ for which M has a constant sign on $\Omega(E_0)$. Theorem 1.1 then determines the monotonicity of the period function on the interval $(0, E_0)$.

The monotonicity of the period function has been extensively studied in the literature (see, for instance, [2–17]). This subject plays a fundamental role in the study of bifurcation phenomena [18–21] and in questions related to inverse problems [17, 22–27].

Despite these contributions, most of the available results are restricted to the potential case or to specific subclasses of Hamiltonian systems.

In the potential case, that is, when $G(y) = \frac{1}{2}y^2$, the monotonicity was analyzed in [5, 6]. The particular case $F' = G'$, under additional assumptions, was studied in [4], while more general approaches can be found in [14, 28]. Related results for separable Hamiltonian systems were obtained in [29].

In this setting, the main criterion for studying the monotonicity was established by Chicone (see [6]). Chicone proved that the period function $T(E)$ is increasing on $(0, E_0)$ if $N(x) \geq 0$ for all $x \in K(E_0)$, and decreasing if $N(x) \leq 0$, where

$$K(E_0) = \{x \in \mathbb{R} : F(x) \leq E_0\} = [x_-(E_0), x_+(E_0)],$$

and

$$N(x) = 6F(x)(F''(x))^2 - 3(F'(x))^2F''(x) - 2F(x)F'(x)F'''(x).$$

Theorem 1.1 provides a natural extension of Chicone's criterion to the class of separable Hamiltonian systems of the form $H(x, y) = F(x) + G(y)$. Indeed, if $G(y) = \frac{1}{2}y^2$, then $(G')^2 - 2GG'' = 0$, and consequently $M(x, y) = \frac{1}{2}N(x)y^4$. Since $y^4 > 0$ for $y \neq 0$, the sign of $M(x, y)$ coincides with the sign of $N(x)$. Thus, Theorem 1.1 recovers Chicone's original criterion in the potential case, while providing a unified framework for more general separable Hamiltonian systems.

We also note that the monotonicity of the period function for more general planar systems has been studied using normalizers (see [14, 28]). In particular, these approaches can be applied to systems of the form

$$x' = G(y), \qquad y' = -F(x).$$

More recently, [29] established several criteria for monotonicity in separable Hamiltonian systems. In particular, Proposition 3.1 in that work provides an expression that

is closely related to our result. A detailed comparison between their criterion and Theorem 1.1 is presented in Sect. 4.

2 Proof of Theorem 1.1

We begin by outlining the proof strategy. Let $0 < E_0 \leq E^*$. To establish Theorem 1.1, it suffices to show that the derivative $T'(E)$ is nonnegative (respectively, nonpositive) on $(0, E_0)$ whenever $M(x, y)$ is nonnegative (respectively, nonpositive) on $\Omega(E_0)$.

The proof relies on expressing the period function as an integral, differentiating it with respect to the energy, and analyzing the resulting expression.

Proof Let $0 < E_0 \leq E^*$ be fixed. For each $E \in (0, E_0)$, let $\gamma_E \subset \Omega(E_0)$ denote the periodic orbit of energy E. Since G is even, the phase portrait of system (2) is symmetric with respect to the x-axis. Hence, the period of γ_E can be written as the convergent improper Riemann integral

$$T(E) = 2\int_{x_-(E)}^{x_+(E)} \frac{dx}{G'(y(x))} = 2\int_{x_-(E)}^{x_+(E)} \frac{dx}{G'\big(G^{-1}(E - F(x))\big)},$$

where $x_-(E)$ and $x_+(E)$ are the intersection points of the level curve $H(x, y) = E$ with the x-axis.

Next, we exploit the parity of G to derive a more convenient expression for

$$G'\big(G^{-1}(z)\big), \qquad z = E - F(x),$$

which appears in the denominator of the above integral.

Since G is even and satisfies $G(0) = G'(0) = 0$, $G''(0) > 0$, there exists a smooth function g, with $g(0) = 0$ and $g'(0) > 0$, such that (see [30])

$$G(y) = g\left(\frac{y^2}{2}\right). \tag{3}$$

Differentiating (3), we obtain

$$G'(y) = y\, g'\left(\frac{y^2}{2}\right). \tag{4}$$

Let $y_0 > 0$ be the positive solution of $G(y) = E_0$. Since the origin is a nondegenerate center, the restriction $G : [0, y_0] \longrightarrow [0, E_0]$ is strictly increasing. Consequently, $G'(y) > 0$, $y \in (0, y_0)$. It follows from (4) that

$$g'(u) > 0, \qquad u \in \left(0, \frac{y_0^2}{2}\right].$$

Hence,

$$g : \left[0, \frac{y_0^2}{2}\right] \longrightarrow [0, E_0]$$

is strictly increasing and therefore admits an inverse

$$g^{-1} : [0, E_0] \longrightarrow \left[0, \frac{y_0^2}{2}\right].$$

Moreover,

$$g^{-1}(0) = 0, \qquad (g^{-1})'(z) > 0, \quad z \in (0, E_0].$$

Define

$$f(z) := \left(\int_0^1 (g^{-1})'(tz)\, dt\right)^{1/2}, \qquad z \in [0, E_0].$$

Since $g^{-1}(0) = 0$, the Fundamental Theorem of Calculus yields

$$g^{-1}(z) = \int_0^z (g^{-1})'(s)\, ds = z \int_0^1 (g^{-1})'(tz)\, dt,$$

and therefore

$$g^{-1}(z) = z f(z)^2. \tag{5}$$

Now let $z = G(y)$. By (3), $g^{-1}(z) = \frac{y^2}{2}$, and hence

$$G^{-1}(z) = y = \sqrt{2\, g^{-1}(z)} = \sqrt{2z}\, f(z).$$

Substituting this identity into (4), we obtain

$$\begin{aligned} G'\big(G^{-1}(z)\big) &= G^{-1}(z)\, g'\left(\frac{(G^{-1}(z))^2}{2}\right) \\ &= G^{-1}(z)\, g'\big(g^{-1}(z)\big) \\ &= \sqrt{2z}\, f(z)\, g'\big(g^{-1}(z)\big). \end{aligned}$$

Recalling that $z = E - F(x)$, the period function becomes

$$T(E) = 2 \int_{x_-(E)}^{x_+(E)} \frac{dx}{\sqrt{2(E - F(x))}\, f(E - F(x))\, g'\big(g^{-1}(E - F(x))\big)}.$$

To compute $T'(E)$, we first introduce the change of variables $r = h(x)$, where

$$h(x) = \begin{cases} x \left(\dfrac{F(x)}{x^2}\right)^{1/2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Since F has a nondegenerate quadratic minimum at the origin, h extends smoothly to $x = 0$. Moreover, one verifies that $h'(x) > 0$ for all x, so that h is a diffeomorphism onto its image. Therefore,

$$T(E) = 2\int_{-\sqrt{E}}^{\sqrt{E}} \frac{dr}{\sqrt{2(E-r^2)}\, f(E-r^2)\, g'\big(g^{-1}(E-r^2)\big)\, h'\big(h^{-1}(r)\big)}.$$

Next, we perform the trigonometric substitution

$$r = \sqrt{E}\sin\theta, \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

This gives

$$\begin{aligned} T(E) &= 2\int_{-\pi/2}^{\pi/2} \frac{\sqrt{E}\cos\theta\, d\theta}{\sqrt{2E\cos^2\theta}\, f(E\cos^2\theta)\, g'\big(g^{-1}(E\cos^2\theta)\big)\, h'\big(h^{-1}(\sqrt{E}\sin\theta)\big)} \\ &= \frac{2}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \frac{d\theta}{h'\big(h^{-1}(\sqrt{E}\sin\theta)\big)\, g'\big(g^{-1}(E\cos^2\theta)\big)\, f(E\cos^2\theta)}. \end{aligned}$$

For every $E \in (0, E_0)$, the integrand is smooth on $(0, E_0) \times [-\pi/2, \pi/2]$. Therefore, differentiation under the integral sign is justified, yielding

$$\begin{aligned} \frac{dT}{dE} &= \frac{2}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \frac{\partial}{\partial E}\left(\frac{1}{h'\big(h^{-1}(\sqrt{E}\sin\theta)\big)\, g'\big(g^{-1}(E\cos^2\theta)\big)\, f(E\cos^2\theta)}\right) d\theta \\ &= -\frac{2}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \frac{\dfrac{\partial A}{\partial E}(E,\theta)}{A(E,\theta)^2}\, d\theta, \end{aligned}$$

where

$$A(E,\theta) = h'\Big(h^{-1}(\sqrt{E}\sin\theta)\Big)\; g'\Big(g^{-1}(E\cos^2\theta)\Big)\; f(E\cos^2\theta).$$

Differentiating $A(E,\theta)$ with respect to E, we obtain

$$\begin{aligned} \frac{\partial A}{\partial E}(E,\theta) = \Bigg\{ &\frac{h''\Big(h^{-1}(\sqrt{E}\sin\theta)\Big)\; g'\big(g^{-1}(E\cos^2\theta)\big)\; f(E\cos^2\theta)\; \sin\theta}{2\sqrt{E}\, h'\Big(h^{-1}(\sqrt{E}\sin\theta)\Big)} \\ &+ h'\Big(h^{-1}(\sqrt{E}\sin\theta)\Big)\left[\frac{g''\big(g^{-1}(E\cos^2\theta)\big)\; f(E\cos^2\theta)\; \cos^2\theta}{g'\big(g^{-1}(E\cos^2\theta)\big)}\right. \\ &\left. + g'\Big(g^{-1}(E\cos^2\theta)\Big)\; f'(E\cos^2\theta)\; \cos^2\theta\right]\Bigg\}. \end{aligned}$$

We decompose this expression as

$$\frac{dT}{dE} = -\frac{2}{\sqrt{2}}\left(I_1(E) + I_2(E) + I_3(E)\right).$$

We first simplify $I_1(E)$ by introducing the factor $\cos^2\theta$ into the integrand. Substituting

$$x = h^{-1}(\sqrt{E}\sin\theta), \qquad z = E - F(x),$$

into $I_1(E)$, we obtain

$$I_1(E) = \frac{1}{2\sqrt{E}}\int_{-\pi/2}^{\pi/2} \frac{h''(x)\sin\theta\, d\theta}{(h'(x))^3\, g'\big(g^{-1}(z)\big)\, f(z)}.$$

Integrating $I_1(E)$ by parts yields

$$I_1(E) = -\frac{1}{2\sqrt{E}}\left[\frac{h''(x)\cos\theta}{(h'(x))^3\, g'\big(g^{-1}(z)\big)\, f(z)}\right]_{-\pi/2}^{\pi/2} + \frac{1}{2\sqrt{E}}\int_{-\pi/2}^{\pi/2}\frac{d}{d\theta}\left[\frac{h''(x)}{(h'(x))^3\, g'\big(g^{-1}(z)\big)\, f(z)}\right]\cos\theta\, d\theta.$$

The boundary term vanishes because $\cos(\pm\pi/2) = 0$. For any smooth function $\Phi(x)$,

$$\frac{d}{d\theta}\Phi(x) = \Phi'(x)\frac{dx}{d\theta}, \qquad \frac{dx}{d\theta} = \frac{\sqrt{E}\cos\theta}{h'(x)}.$$

Define

$$B = B(E,\theta) := \frac{h''(x)}{(h'(x))^3\, g'\big(g^{-1}(z)\big)\, f(z)}.$$

Using the chain rule,

$$\frac{dB}{d\theta} = \frac{h'''\frac{dx}{d\theta}(h')^3\, g'\big(g^{-1}(z)\big) f(z) - h''\frac{d}{d\theta}\big[(h')^3\, g'\big(g^{-1}(z)\big) f(z)\big]}{\big[(h')^3\, g'\big(g^{-1}(z)\big) f(z)\big]^2}.$$

Moreover,

$$\begin{aligned}
&\frac{d}{d\theta}\Big[(h')^3 g'\big(g^{-1}(z)\big) f(z)\Big] \\
&= 3(h')^2 h''\frac{dx}{d\theta} g'\big(g^{-1}(z)\big) f(z) \\
&\quad + (h')^3\left[\frac{g''\big(g^{-1}(z)\big)(-F'(x))\frac{dx}{d\theta}}{g'\big(g^{-1}(z)\big)} f(z) + g'\big(g^{-1}(z)\big) f'(z)(-F'(x))\frac{dx}{d\theta}\right]
\end{aligned}$$

$$= \frac{dx}{d\theta}\frac{1}{g'\big(g^{-1}(z)\big)}\Big\{3(h')^2h''\big[g'\big(g^{-1}(z)\big)\big]^2 f(z) - (h')^3 g''\big(g^{-1}(z)\big)F'(x)f(z) - (h')^3\big[g'\big(g^{-1}(z)\big)\big]^2 f'(z)F'(x)\Big\}.$$

Substituting this expression into the previous identity gives

$$\frac{dB}{d\theta} = \frac{\frac{dx}{d\theta}}{(h')^6\big(g'\big(g^{-1}(z)\big)\big)^3(f(z))^2}\Big\{h'''(h')^3\big[g'\big(g^{-1}(z)\big)\big]^2 f(z) - 3(h'')^2(h')^2\big[g'\big(g^{-1}(z)\big)\big]^2 f(z) + h''(h')^3 g''\big(g^{-1}(z)\big)F'(x)f(z) + h''(h')^3\big[g'\big(g^{-1}(z)\big)\big]^2 F'(x)f'(z)\Big\}.$$

Hence,

$$I_1(E) = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\frac{\cos^2\theta\, d\theta}{(h')^5\big(g'\big(g^{-1}(z)\big)\big)^3(f(z))^2}\Big\{h'''h'\big[g'\big(g^{-1}(z)\big)\big]^2 f(z) - 3(h'')^2\big[g'\big(g^{-1}(z)\big)\big]^2 f(z) + h''h' g''\big(g^{-1}(z)\big)F'(x)f(z) + h''h'\big[g'\big(g^{-1}(z)\big)\big]^2 F'(x)f'(z)\Big\}.$$

We rewrite $I_2(E)$ and $I_3(E)$ with the same denominator as $I_1(E)$:

$$I_2(E) = \int_{-\pi/2}^{\pi/2}\frac{g''\big(g^{-1}(z)\big)\cos^2\theta}{h'(x)\big(g'\big(g^{-1}(z)\big)\big)^3 f(z)}\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{(h')^4 g''\big(g^{-1}(z)\big)f(z)\cos^2\theta}{(h')^5\big(g'\big(g^{-1}(z)\big)\big)^3(f(z))^2}\,d\theta,$$

$$I_3(E) = \int_{-\pi/2}^{\pi/2}\frac{f'(z)\cos^2\theta}{h'(x)g'\big(g^{-1}(z)\big)(f(z))^2}\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{(h')^4\big(g'\big(g^{-1}(z)\big)\big)^2 f'(z)\cos^2\theta}{(h')^5\big(g'\big(g^{-1}(z)\big)\big)^3(f(z))^2}\,d\theta.$$

Combining the above identities yields

$$\frac{dT}{dE}$$

$$
\begin{aligned}
&= -\frac{2}{\sqrt{2}}\big(I_1(E) + I_2(E) + I_3(E)\big) \\
&= \frac{1}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \Big\{ 3(h'')^2 \left[g'\big(g^{-1}(z)\big)\right]^2 f(z) - h''' h' \left[g'\big(g^{-1}(z)\big)\right]^2 f(z) \\
&\quad - h'' h' \, g''\big(g^{-1}(z)\big) F'(x) f(z) - h'' h' \left[g'\big(g^{-1}(z)\big)\right]^2 F'(x) f'(z) \\
&\quad - 2(h')^4 \, g''\big(g^{-1}(z)\big) f(z) - 2(h')^4 \left[g'\big(g^{-1}(z)\big)\right]^2 f'(z) \Big\} \\
&\frac{\cos^2\theta}{(h')^5 \left(g'\big(g^{-1}(z)\big)\right)^3 (f(z))^2} \, d\theta.
\end{aligned}
$$

Since the denominator is positive, the sign of $T'(E)$ is determined by the sign of

$$
\begin{aligned}
K = &\left[3(h'')^2 - h''' h'\right]\big(g'(g^{-1}(z))\big)^2 f(z) - \left[h'' h' F'(x) + 2(h')^4\right] \\
&\Big[\big(g'(g^{-1}(z))\big)^2 f'(z) + g''(g^{-1}(z)) f(z)\Big].
\end{aligned}
$$

Since $h(x)^2 = F(x)$, we have

$$
h' = \frac{F'}{2h}, \qquad h'' = \frac{2F''F - (F')^2}{4hF}, \qquad h''' = \frac{4F^2F''' - 6FF'F'' + 3(F')^3}{8hF^2}.
$$

Since h is smooth, the above identities, initially derived for $x \neq 0$, also hold at $x = 0$. Consequently,

$$
3(h'')^2 - h''' h' = \frac{6F(F'')^2 - 3(F')^2F'' - 2FF'F'''}{8F^2},
$$

and

$$
h'' h' F' + 2(h')^4 = \frac{2F(F')^2F''}{8F^2}.
$$

Recall that along the periodic orbit $H(x, y) = E, z = E - F(x) = G(y)$. Therefore, all the quantities involving g and f below are evaluated at $z = G(y)$. Since

$$
g^{-1}(G(y)) = \frac{y^2}{2},
$$

it follows from (4) and (5) that

$$
g'\big(g^{-1}(G(y))\big) = g'\left(\frac{y^2}{2}\right) = \frac{G'(y)}{y},
$$

and

$$
f(G(y))^2 = \frac{g^{-1}(G(y))}{G(y)} = \frac{y^2}{2\,G(y)}.
$$

Consequently,

$$\left(g'(g^{-1}(G(y)))\right)^2 f(G(y))^2 = \frac{(G'(y))^2}{2\,G(y)},$$

$$\left(g'(g^{-1}(G(y)))\right)^2 f(G(y))\, f'(G(y)) = \frac{2\,G(y)G''(y) - (G'(y))^2}{4\,G(y)^2},$$

and

$$g''\left(g^{-1}(G(y))\right) f(G(y))^2 = \frac{G''(y) - \dfrac{G'(y)}{y}}{2\,G(y)}.$$

Combining the last two identities, we obtain

$$\begin{aligned}&\left(g'(g^{-1}(G(y)))\right)^2 f(G(y))\, f'(G(y)) + g''\left(g^{-1}(G(y))\right) f(G(y))^2\\ &= \frac{2G(y)G''(y) - (G'(y))^2}{4G(y)^2}.\end{aligned}$$

Multiplying K by the positive quantity $f(z) = f(G(y))$, we obtain

$$\begin{aligned}K \cdot f &= \left[3(h'')^2 - h'''h'\right]\left(g'(g^{-1})\right)^2 f^2 - \left[h''h'F' + 2(h')^4\right]\left[\left(g'(g^{-1})\right)^2 ff' + g''(g^{-1})f^2\right]\\ &= \left[3(h'')^2 - h'''h'\right]\frac{(G')^2}{2G} - \left[h''h'F' + 2(h')^4\right]\frac{2GG'' - (G')^2}{4G^2}\\ &= \left[\frac{6F(F'')^2 - 3(F')^2F'' - 2FF'F'''}{8F^2}\right]\frac{(G')^2}{2G} - \left[\frac{2F(F')^2F''}{8F^2}\right]\frac{2GG'' - (G')^2}{4G^2}\\ &= \frac{\left(6F(F'')^2 - 3(F')^2F'' - 2FF'F'''\right)G(G')^2 - F(F')^2F''\left(2GG'' - (G')^2\right)}{16F^2G^2}.\end{aligned}$$

Therefore, the sign of $T'(E)$ is determined by the sign of

$$M(x, y) = \left(6F(F'')^2 - 3(F')^2F'' - 2FF'F'''\right)G(G')^2 + F(F')^2F''\left((G')^2 - 2GG''\right).$$

Consequently, if $M(x, y) \geq 0$ for all $(x, y) \in \Omega(E_0)$, then $T'(E) \geq 0$ for all $E \in (0, E_0)$. Likewise, if $M(x, y) \leq 0$ for all $(x, y) \in \Omega(E_0)$, then $T'(E) \leq 0$ for all $E \in (0, E_0)$.

This completes the proof of Theorem 1.1. □

3 Applications

In this section, we illustrate the applicability of Theorem 1.1 through several examples, including polynomial, trigonometric, hyperbolic, and relativistic Hamiltonian systems.

3.1 A Class of Polynomial Hamiltonian Systems

Example 3.1 Consider the Hamiltonian system

$$\begin{cases} x_1' = x_2\big(b_1 + b_2 x_2^2\big), \\ x_2' = -x_1\big(a_1 + a_2 x_1 + a_3 x_1^2\big), \end{cases} \tag{6}$$

where $a_1 > 0$ and $b_1 > 0$.

By the change of variables

$$x = x_1, \qquad y = \sqrt{\frac{b_1}{a_1}}\, x_2, \qquad d\tau = \sqrt{a_1 b_1}\, dt, \tag{7}$$

we obtain the normalized system

$$\begin{cases} x' = y\big(1 + cy^2\big), \\ y' = -x\big(1 + ax + bx^2\big), \end{cases} \tag{8}$$

where

$$a = \frac{a_2}{a_1}, \qquad b = \frac{a_3}{a_1}, \qquad c = \frac{a_1 b_2}{b_1^2}.$$

The Hamiltonian function corresponding to system (8) is

$$H(x, y) = \frac{1}{2}x^2 + \frac{a}{3}x^3 + \frac{b}{4}x^4 + \frac{1}{2}y^2 + \frac{c}{4}y^4.$$

Hence,

$$F(x) = \frac{1}{2}x^2 + \frac{a}{3}x^3 + \frac{b}{4}x^4, \qquad G(y) = \frac{1}{2}y^2 + \frac{c}{4}y^4.$$

A direct computation shows that

$$M(x, y) = \frac{1}{24}x^4 y^4 \big(A(x)P(y) - B(x)Q(y)\big),$$

where

$$\begin{aligned} A(x) &= 10a^2 - 9b + a(30b + 4a^2)x + b(36b + 16a^2)x^2 + 24ab^2x^3 + 9b^3x^4, \\ B(x) &= (6 + 4ax + 3bx^2)(1 + ax + bx^2)^2(1 + 2ax + 3bx^2), \\ P(y) &= (1 + cy^2)^2(2 + cy^2), \\ Q(y) &= c(3 + cy^2). \end{aligned}$$

Since $M(x, y)$ and

$$N(x, y) = A(x)P(y) - B(x)Q(y)$$

have the same sign, the monotonicity of the period function $T(E)$ is determined by the sign of $N(x, y)$.

We also observe that

$$A(-x, -a, b) = A(x, a, b), \quad B(-x, -a, b) = B(x, a, b),$$
$$F(-x, -a, b) = F(x, a, b).$$

Hence, the transformation

$$(x, a) \longmapsto (-x, -a)$$

preserves the functions A, B, and F, and maps the projection of the period annulus corresponding to the parameters (a, b) onto that corresponding to $(-a, b)$. This symmetry will be used repeatedly in the proofs to reduce the arguments for the cases $a < 0$ to the corresponding ones with $a > 0$.

The following technical lemma will be used repeatedly in the proof of Cases (A) and (B).

Lemma 3.2 *Let I be the projection of the period annulus of the center onto the x-axis. Suppose that*

$$A(x) > 0, \qquad x \in I,$$

and that there exists $\xi \in I$ such that

$$\frac{B(x)}{A(x)} \leq \frac{B(\xi)}{A(\xi)}, \qquad x \in I.$$

Define

$$c_0 := \frac{2A(\xi)}{3B(\xi)}.$$

Then,

$$N(x, y) \geq 0$$

throughout the period annulus whenever $0 \leq c \leq c_0$.

Proof Since

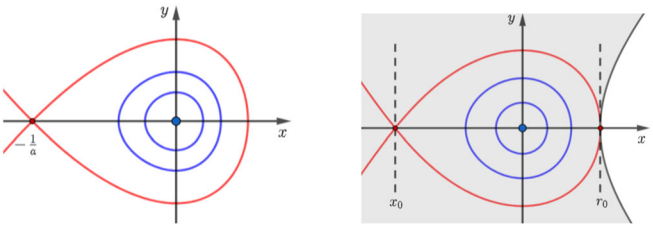
$$P(y) = (1 + cy^2)^2(2 + cy^2) > 0,$$

the inequality $N(x, y) \geq 0$ is equivalent to

$$A(x) - B(x)\frac{Q(y)}{P(y)} \geq 0.$$

Moreover, the function $\frac{Q(y)}{P(y)}$ is strictly decreasing with respect to y^2. Hence,

$$\frac{Q(y)}{P(y)} \leq \frac{Q(0)}{P(0)} = \frac{3c}{2} \leq \frac{3c_0}{2} = \frac{A(\xi)}{B(\xi)}.$$



(a) $\Omega_a(E_0)$ for $a > 0$. (b) $M(x, y) \geq 0$ in $\Omega_a(E_0)$.

Fig. 2 Illustration of Case (A)

Therefore,

$$A(x) - B(x)\frac{Q(y)}{P(y)} \geq A(x) - \frac{A(\xi)}{B(\xi)}B(x).$$

Finally, the hypothesis

$$\frac{B(x)}{A(x)} \leq \frac{B(\xi)}{A(\xi)}$$

is equivalent to

$$A(x) - \frac{A(\xi)}{B(\xi)}B(x) \geq 0.$$

Consequently, $N(x, y) \geq 0$ throughout the period annulus. □

We now analyze the different polynomial families by verifying the assumptions of Lemma 3.2 in each case.

Case (A): $a \neq 0, b = 0, c \geq 0$.

In this case, the origin $(0, 0)$ is a bounded center contained in the region

$$\Omega(E^*) = \{(x, y) : 0 \leq H(x, y) \leq E^*\},$$

where $E^* = F(x_0)$ and $x_0 = -\frac{1}{a}$ is the root of $1 + ax = 0$ (see Fig. 2a).

Let

$$r_0 = \frac{1}{2a}$$

be the second intersection of the energy level $F(x) = F(x_0)$ with the x-axis, and define

$$c_0 := \frac{2A(r_0)}{3B(r_0)} = \frac{2a^2}{9}.$$

We have the following result.

Lemma 3.3 *Suppose that $a \neq 0$, $b = 0$, and $0 \leq c \leq c_0$. Then the period function is increasing on the entire period annulus.*

Proof In this case,

$$A(x) = a^2(10 + 4ax), \qquad B(x) = (6 + 4ax)(1 + ax)^2(1 + 2ax).$$

By the symmetry reduction, it suffices to assume that $a > 0$. Since

$$A(x) = a^2(10 + 4ax) > 0, \qquad x \in [x_0, r_0],$$

the function

$$f(x) = \frac{B(x)}{A(x)}$$

is well defined on I. Moreover,

$$f'(x) > 0, \qquad x \in (x_0, r_0),$$

and therefore

$$\frac{B(x)}{A(x)} \leq \frac{B(r_0)}{A(r_0)}, \qquad x \in [x_0, r_0].$$

Since $I \subset [x_0, r_0]$, the above inequality also holds for every $x \in I$. Finally,

$$c_0 = \frac{2A(r_0)}{3B(r_0)} = \frac{2a^2}{9},$$

so all the hypotheses of Lemma 3.2 are satisfied. Therefore, $N(x, y) \geq 0$ throughout the period annulus. Consequently, $M(x, y) \geq 0$ throughout the period annulus, and the conclusion follows from Theorem 1.1. □

Case (B): $b \neq 0$ and $\Delta = a^2 - 4b \geq 0$.

In this case, the polynomial

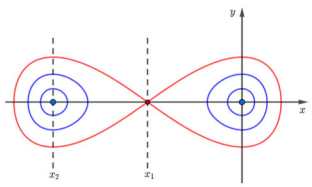
$$p(x) = 1 + ax + bx^2$$

has two real roots, denoted by x_1 and x_2. Consequently, the origin $(0, 0)$ is a bounded center contained in the region

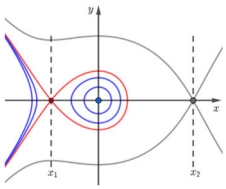
$$\Omega(E^*) = \{(x, y) : 0 \leq H(x, y) \leq E^*\},$$

where $E^* = F(x_0)$ and $x_0 \in \{x_1, x_2\}$ is the root of $p(x) = 0$ that determines the homoclinic orbit bounding the period annulus. Without loss of generality, we assume that $x_0 = x_1$ (see Fig. 3).

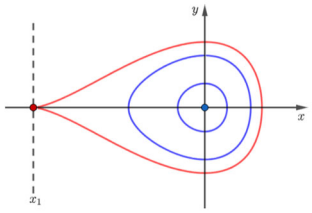
We distinguish the following subcases.



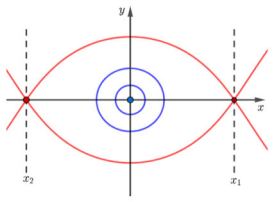
(a) $x_2 < x_1 < 0$ $(a > 0,\ b > 0)$.



(b) $x_1 < 0 < x_2$ $(a > 0,\ b < 0)$.



(c) $x_1 = x_2$ $(a > 0,\ b = a^2/4)$.



(d) $x_2 < 0,\ x_1 = -x_2$ $(a = 0,\ b < 0)$.

Fig. 3 Four of the seven possible phase configurations

Subcase (B1): $a = 0, b < 0, c \geq 0$.

Let

$$x_0 = -\frac{1}{\sqrt{-b}}, \qquad r_0 = \frac{1}{\sqrt{-b}},$$

where $F(r_0) = F(x_0)$, and define

$$c_0 := \frac{2A(0)}{3B(0)} = -b.$$

We have the following result.

Lemma 3.4 *Suppose that $a = 0$, $b < 0$, and $0 \leq c \leq c_0$. Then the period function is increasing on the entire period annulus.*

Proof In this case,

$$A(x) = 9b\left(-1 + 4bx^2 + b^2x^4\right)$$

and

$$B(x) = (6 + 3bx^2)(1 + bx^2)^2(1 + 3bx^2).$$

Since $b < 0$,

$$A(x) > 0, \qquad x \in [x_0, r_0].$$

Hence,

$$f(x) = \frac{B(x)}{A(x)}$$

is well defined on $[x_0, r_0]$.

Moreover, A and B are even functions, and therefore so is f. Since f attains its global maximum on $[x_0, r_0]$ at $x = 0$,

$$\frac{B(x)}{A(x)} \leq \frac{B(0)}{A(0)}, \qquad x \in [x_0, r_0].$$

Finally,

$$c_0 = \frac{2A(0)}{3B(0)} = -b,$$

so all the hypotheses of Lemma 3.2 are satisfied. Therefore, $M(x, y) \geq 0$ throughout the period annulus, and the conclusion follows from Theorem 1.1. □

Subcase (B2): $a \neq 0, b = \frac{a^2}{4}, c \geq 0$.

Let

$$x_0 = -\frac{2}{a}, \qquad r_0 = \frac{2}{3a},$$

where $F(r_0) = F(x_0)$, and define

$$c_0 := \frac{2A(r_0)}{3B(r_0)} = \frac{21a^2}{128}.$$

We have the following result.

Lemma 3.5 *Suppose that $a \neq 0$, $b = \frac{a^2}{4}$, and $0 \leq c \leq c_0$. Then the period function is increasing on the entire period annulus.*

Proof In this case,

$$A(x) = \frac{1}{64}(ax + 2)^2(9a^2x^2 + 60ax + 124).$$

Since $A(x_0) = 0$ and

$$A(x) > 0, \qquad x \in (x_0, r_0),$$

the function

$$f(x) = \frac{B(x)}{A(x)}$$

is well defined on the projection I of the period annulus, where $I = (x_0, r_0)$.

Moreover, f attains its maximum on I at $x = r_0$. Hence,

$$\frac{B(x)}{A(x)} \leq \frac{B(r_0)}{A(r_0)}, \qquad x \in I.$$

Finally,

$$c_0 = \frac{2A(r_0)}{3B(r_0)} = \frac{21a^2}{128},$$

so all the hypotheses of Lemma 3.2 are satisfied. Consequently, $M(x, y) \geq 0$ throughout the period annulus, and the conclusion follows from Theorem 1.1. □

Subcase (B3): $a \neq 0, b \neq 0, \Delta = a^2 - 4b > 0, c \geq 0$.

Let x_0 be the negative root of

$$p(x) = 1 + ax + bx^2,$$

and let $r_0 > x_0$ be the second solution of $F(r_0) = F(x_0)$. Unlike the previous subcases, the proof that

$$A(x) > 0, \qquad x \in [x_0, r_0],$$

is not immediate. For completeness, it is established in Appendix A(Lemma A.1). Consequently,

$$f(x) = \frac{B(x)}{A(x)}$$

is well defined on the projection of the period annulus. Since f is continuous on $[x_0, r_0]$, there exists $r \in [x_0, r_0]$ such that

$$f(r) = \max_{x \in [x_0, r_0]} f(x).$$

Moreover,

$$f(0) = \frac{B(0)}{A(0)} = \frac{6}{10a^2 - 9b} > 0,$$

so

$$f(r) \geq f(0) > 0.$$

Since $A(r) > 0$, it follows that $B(r) > 0$. Therefore,

$$c_0 := \frac{2A(r)}{3B(r)}$$

is well defined. We have the following result.

Lemma 3.6 *Suppose that $a \neq 0$, $b \neq 0$, $\Delta = a^2 - 4b > 0$, and $0 \leq c \leq c_0$. Then the period function is increasing on the entire period annulus.*

Proof Since $f(x) \leq f(r)$ for all $x \in [x_0, r_0]$, all the assumptions of Lemma 3.2 are satisfied. Therefore, $M(x, y) \geq 0$ throughout the period annulus, and the conclusion follows from Theorem 1.1. □

Remark 3.7 The following observations apply to Cases (A) and (B).

1. Let $c_1 = \dfrac{2A(0)}{3B(0)}$. For each fixed $c \in (c_0, c_1)$, there exists $x_c \in (x_0, r_0)$ such that

$$N(x, y) \geq 0, \quad \text{for all } (x, y) \text{ satisfying } 0 \leq H(x, y) \leq F(x_c) \text{ (see Fig. 4a).}$$

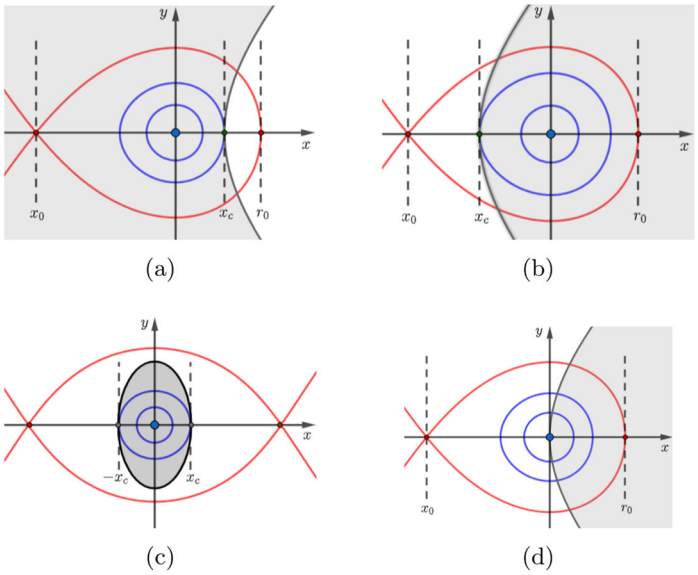


Fig. 4 Sign of $N(x, y)$ in the region $\{(x, y) : 0 \leq H(x, y) \leq F(x_c)\}$

Numerical evidence suggests that the period function $T(E)$ attains a maximum at some $E_* \in (F(x_c), F(x_0))$, and becomes decreasing on the interval $(E_*, F(x_0))$.

2. For each fixed $c > c_1$, there exists $x_c \in (x_0, r_0)$ such that

$$N(x, y) \leq 0, \quad \text{for all } (x, y) \text{ satisfying } 0 \leq H(x, y) \leq F(x_c) \text{ (see Fig. 4b).}$$

Consequently, $T(E)$ is decreasing on $(0, F(x_c)]$. Numerical experiments suggest that $T(E)$ attains a minimum at some $E_* \in (F(x_c), F(x_0))$, and becomes increasing on $(E_*, F(x_0))$.

3. In the special case $a = 0$ and $b < 0$, one has $c_0 = c_1$. Hence, $T(E)$ is increasing on $(0, F(x_0)]$ for all $0 \leq c \leq c_0$, and decreasing on $(0, F(x_c)]$ for each fixed $c > c_0$ (see Fig. 4c).
4. When $a \neq 0$ and $c = c_0$, the sign of $N(x, y)$ is not well defined in a neighborhood of $(0, 0)$. Therefore, the criterion does not provide a definite conclusion regarding the monotonicity of the period function near the origin (see Fig. 4d).
5. In the cases $b = 0$ (Case (A)) or $b > 0$, the point x_c can be characterized as a solution of the equation

$$\sigma(x_c) = 0, \quad \text{where} \quad \sigma(x) = 2A(x) - 3cB(x).$$

Example 3.8 *(A Hamiltonian considered in [14])* The Hamiltonian system

$$H(x, y) = k\left(\frac{x^2}{2} + \frac{x^3}{3}\right) + \frac{y^2}{2} + \frac{y^4}{4},$$

was considered by Freire, Gasull and Guillamon [14] as an example illustrating the limitations of their monotonicity criterion. Since this Hamiltonian corresponds to an I-D pair, Proposition 10 of [14] does not determine the monotonicity of its period function. Nevertheless, the numerical computations reported in Fig. 2 of [14] indicate that the period function is increasing for $k = 1$, nearly isochronous for $k \approx 1.17525$, and decreasing for larger values of k.

After applying the change of variables (7), the Hamiltonian system is transformed into

$$x' = y(1 + ky^2), \qquad y' = -x(1 + x),$$

which is precisely Case (A) with parameter $c = k$. Hence, the results established above apply directly.

In this case,

$$x_0 = -1, \qquad F(x_0) = \frac{1}{6},$$

and the equation $F(r_0) = F(x_0)$ has the solution $r_0 = \frac{1}{2}$. Since

$$A(x) = 10 + 4x, \qquad B(x) = (6 + 4x)(1 + x)^2(1 + 2x),$$

we obtain

$$k_0 = \frac{2A(1/2)}{3B(1/2)} = \frac{2}{9}.$$

Therefore, by Lemma 3.3, the period function $T(E)$ is increasing on $(0, 1/6)$ for every

$$0 < k \leq \frac{2}{9}.$$

Moreover,

$$k_1 = \frac{2A(0)}{3B(0)} = \frac{10}{9}.$$

Hence, by Remark 3.7(1), for every

$$\frac{2}{9} < k < \frac{10}{9},$$

there exists $0 < x_k < \frac{1}{2}$ such that $T(E)$ is increasing on $(0, F(x_k))$. For instance, when $k = 1$,

$$x_1 \approx 0.025147, \qquad F(x_1) \approx 0.000321,$$

so that $T(E)$ is increasing on $(0, 0.000321)$. This is consistent with the numerical computations reported in [14], which indicate that the period function remains increasing on the whole interval $(0, 1/6)$.

For the limiting value $k = \frac{10}{9}$, our criterion is inconclusive. However, the numerical computations of [14] show that the period function is nearly constant near the origin, giving the appearance of an almost isochronous center.

Finally, when

$$k > \frac{10}{9},$$

Remark 3.7(2) implies that $T(E)$ is decreasing on

$$0 < E \leq F(x_k) < \frac{1}{6}.$$

For example, when $k = 2$,

$$x_2 \approx -0.12449, \qquad F(x_2) \approx 0.007106,$$

and therefore $T(E)$ is decreasing on $(0, 0.007106)$. The numerical computations reported in [14] indicate that the period function subsequently attains a minimum at approximately

$$E_* \approx 0.0427,$$

after which it increases. A similar behavior is observed for $k = 10$ and larger values of k.

Therefore, although the criterion of [14] does not determine the monotonicity of the period function for this Hamiltonian, the present approach provides rigorous analytical information on explicit energy intervals. These analytical results are in full qualitative agreement with the numerical computations reported in Fig. 2 of [14].

Case (C): $\Delta < 0$

In this case, the period annulus of the origin coincides with $\mathbb{R}^2\backslash\{(0, 0)\}$. Unlike the case $\Delta \geq 0$, the function N does not preserve a constant sign throughout the period annulus. The following results provide sufficient conditions for the monotonicity of the period function on an interval of energies $(0, E_0)$, where E_0 depends on the parameters and is chosen so that

$$\Omega(E_0) = \{(x, y) \in \mathbb{R}^2 : 0 \leq H(x, y) \leq E_0\}$$

is contained in a region where $N(x, y)$ preserves a constant sign. The optimal choice of E_0 would correspond to the first energy level for which the level curve $H(x, y) = E_0$ is tangent to the curve $N(x, y) = 0$. In the following subcases, we determine explicit values of E_0 for which N preserves a constant sign on $\Omega(E_0)$.

We first observe that the function B is strictly positive on the entire real line. Indeed, writing

$$B(x) = B_0(x)B_1(x)B_2(x),$$

where

$$B_0(x) = \left(1 + ax + bx^2\right)^2, \qquad B_1(x) = 1 + 2ax + 6bx^2,$$

and

$$B_2(x) = 6 + 4ax + 3bx^2,$$

we have

$$\Delta < 0, \qquad \Delta_1 = 4(\Delta - 2b) < 0, \qquad \Delta_2 = 8(2\Delta - b) < 0.$$

Hence, B_0, B_1, and B_2 have no real zeros. Since

$$B_0(0) = 1, \qquad B_1(0) = 1, \qquad B_2(0) = 6,$$

each of these polynomials is positive on $\mathbb{R}$. Therefore, $B(x) > 0$, for all $x \in \mathbb{R}$.

We now consider the different subcases (Fig. 5).

Subcase (C1): $\Delta < 0$, $a = 0$.

Let

$$x_0 = \sqrt{\frac{\sqrt{5}-2}{b}}, \qquad E_0 := F(x_0).$$

We have the following result.

Lemma 3.9 *Suppose $\Delta < 0$, $a = 0$, and $c \geq 0$. Then:*

(i) If $c = 0$, then $N(x, y) \leq 0$ in $\Omega(E_0)$. Consequently, the period function is decreasing on $(0, E_0)$.

(ii) Assume that $c > 0$. Let $x_m > x_0$ be a point at which

$$f(x) = \frac{A(x)}{B(x)}$$

attains its absolute maximum, and define

$$E_0 := F(x_m).$$

Then there exists a unique constant $c_0 > 0$ such that

$$\frac{A(x_m)}{B(x_m)} = \frac{Q(y_m)}{P(y_m)},$$

where $y_m = y_m(c_0)$ is determined by

$$G(y_m) = E_0.$$

Moreover, for every $c \geq c_0$,

$$N(x, y) \leq 0 \quad \text{in } \Omega(E_0).$$

Consequently, the period function is decreasing on $(0, E_0)$.

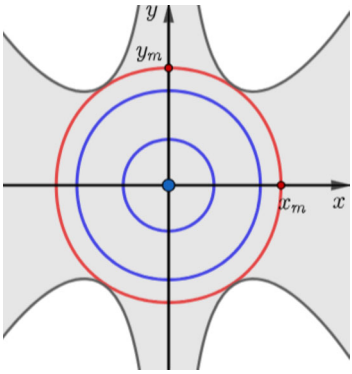


Fig. 5 Region $\Omega(E_0)$ corresponding to Case (C1)(ii)

Proof We work under the assumptions $a = 0$, $b > 0$, and $c \geq 0$. In this case,

$$A(x) = 9b^3x^4 + 36b^2x^2 - 9b = 9b\big(b^2x^4 + 4bx^2 - 1\big).$$

(i) Suppose $c = 0$. Then $P(y) = 4$, $Q(y) = 0$, and therefore $N(x, y) = 4A(x)$. The function $A(x)$ vanishes at

$$x = \pm x_0, \qquad x_0 = \sqrt{\frac{\sqrt{5} - 2}{b}},$$

and satisfies

$$A(x) \leq 0, \qquad |x| \leq x_0.$$

Hence, $N(x, y) \leq 0$ throughout $\Omega(E_0)$. By Theorem 1.1, the period function is decreasing on $(0, E_0)$.

(ii) Assume that $c > 0$. Since $B(x) > 0$ and $P(y) > 0$,

$$N(x, y) \leq 0 \iff \frac{A(x)}{B(x)} \leq \frac{Q(y)}{P(y)}.$$

Define

$$f(x) = \frac{A(x)}{B(x)}.$$

Since

$$f(x_0) = 0, \qquad \lim_{x \to +\infty} f(x) = 0,$$

there exists $x_m > x_0$ such that

$$f(x_m) = \max_{x \geq x_0} f(x) > 0.$$

Let $E_0 := F(x_m)$, and let $y_m = y_m(c)$ be determined by $G(y_m) = E_0$. Define

$$\varphi(c) = \min_{|y| \le y_m(c)} \frac{Q(y)}{P(y)}.$$

Since Q/P is continuous and positive for $c > 0$, it follows that φ is continuous. Therefore, there exists a unique constant $c_0 > 0$ such that

$$\varphi(c_0) = f(x_m).$$

Equivalently,

$$\frac{A(x_m)}{B(x_m)} = \frac{Q(y_m(c_0))}{P(y_m(c_0))}.$$

Hence, for every $c \ge c_0$,

$$f(x_m) \le \varphi(c) \le \frac{Q(y)}{P(y)},$$

whenever $G(y) \le E_0$. Since

$$\frac{A(x)}{B(x)} \le f(x_m),$$

it follows that

$$\frac{A(x)}{B(x)} \le \frac{Q(y)}{P(y)}$$

for all (x, y) satisfying $H(x, y) \le E_0$. Therefore, $N(x, y) \le 0$ throughout $\Omega(E_0)$. By Theorem 1.1, the period function is decreasing on $(0, E_0)$. □

Subcase (C2): $a \neq 0, b > \frac{a^2}{4}, c = 0$.

Lemma 3.10 *Suppose $a \neq 0$, $b > \frac{a^2}{4}$, and $c = 0$. Then the following statements hold.*

(i) If $a > 0$ and $\frac{a^2}{4} < b < \frac{10}{9}a^2$, then $A(x)$ has two negative real roots $x_2 < x_1 < 0$. Let $E_0 := F(x_1)$. Then $N(x, y) \ge 0$ in $\Omega(E_0)$. Consequently, the period function is increasing on $(0, E_0)$.

(ii) If $a < 0$ and $\frac{a^2}{4} < b < \frac{10}{9}a^2$, then $A(x)$ has two positive real roots $0 < x_1 < x_2$. Let $E_0 := F(x_1)$. Then $N(x, y) \ge 0$ in $\Omega(E_0)$. Consequently, the period function is increasing on $(0, E_0)$.

(iii) If $b > \frac{10}{9}a^2$, then $A(x)$ has two real roots $x_2 < 0 < x_1$. Let $E_0 := F(x_1)$. Then $N(x, y) \le 0$ in $\Omega(E_0)$. Consequently, the period function is decreasing on $(0, E_0)$.

(iv) If $a > 0$ and $b = \frac{10}{9}a^2$, then $N(x, y)$ changes sign in every neighborhood of the origin. Consequently, Theorem 1.1 does not provide information on the monotonicity of the period function.

Proof Since $c = 0$, $P(y) = 2$ and $Q(y) = 0$. Therefore $N(x, y) = 2A(x)$. Hence, the sign of $N(x, y)$ coincides with that of $A(x)$. Moreover, $\lim_{x\to\pm\infty} A(x) = +\infty$, since A has leading coefficient $9b^3 > 0$, and

$$A(0) = 10a^2 - 9b, \qquad A'(0) = a(30b + 4a^2).$$

(i) If

$$a > 0, \qquad \frac{a^2}{4} < b < \frac{10}{9}a^2,$$

then $A(0) > 0$. Since $A(x) \to +\infty$ as $x \to -\infty$, continuity implies that A has two negative roots $x_2 < x_1 < 0$. Hence,

$$A(x) \geq 0, \qquad x \in [x_1, 0].$$

Therefore, $N(x, y) \geq 0$ in $\Omega(E_0)$, where $E_0 = F(x_1)$. The conclusion follows from Theorem 1.1.

(ii) If

$$a < 0, \qquad \frac{a^2}{4} < b < \frac{10}{9}a^2,$$

the argument is analogous. In this case, $0 < x_1 < x_2$, and

$$A(x) \geq 0, \qquad x \in [0, x_1].$$

Therefore, $N(x, y) \geq 0$ in $\Omega(E_0)$, where $E_0 = F(x_1)$. The conclusion again follows from Theorem 1.1.

(iii) If $b > \frac{10}{9}a^2$, then $A(0) < 0$. Since $A(x) \to +\infty$ as $x \to \pm\infty$, there exist two real roots $x_2 < 0 < x_1$, and

$$A(x) \leq 0, \qquad x \in [x_2, x_1].$$

Therefore, $N(x, y) \leq 0$ in $\Omega(E_0)$, where $E_0 = F(x_1)$. The conclusion follows from Theorem 1.1.

(iv) If $b = \frac{10}{9}a^2$, then

$$A(0) = 0, \qquad A'(0) \neq 0.$$

Hence A, and consequently N, changes sign at the origin. Therefore, Theorem 1.1 does not provide information on the monotonicity of the period function. □

Subcase (C3): $a \neq 0, c > 0, b > \frac{a^2}{4}$.

Lemma 3.11 *Suppose $a \neq 0$, $b > \dfrac{a^2}{4}$, and $c > 0$. Then the following statements hold.*

(i) Suppose $\frac{a^2}{4} < b \leq \frac{a^2}{3}$. *Then* $A(x)$ *has two distinct real roots* $x_2 < x_1 < 0$, *if* $a > 0$, *and* $0 < x_1 < x_2$, *if* $a < 0$. *Let* $x_0 \neq x_1$ *satisfy* $F(x_0) = F(x_1)$, *and define* $E_0 := F(x_1)$. *Then* $N(x, y) \geq 0$ *in* $\Omega(E_0)$, *for every*

$$0 < c \leq c_0 := \frac{2A(x_0)}{3B(x_0)}.$$

Consequently, the period function is increasing on $(0, E_0)$.

(ii) Suppose $b > \frac{a^2}{3}$. *Let* $f(x) = \frac{A(x)}{B(x)}$, *and let* x_m *be a point at which* f *attains its absolute maximum. Define* $E_0 := F(x_m)$, *and let* $y_m = y_m(c)$ *be determined by*

$$G(y_m) = E_0.$$

Then there exists a unique constant $c_0 > 0$ *such that*

$$\frac{A(x_m)}{B(x_m)} = \frac{Q(y_m)}{P(y_m)}.$$

Moreover, for every $c \geq c_0$, $N(x, y) \leq 0$ *in* $\Omega(E_0)$. *Consequently, the period function is decreasing on* $(0, E_0)$.

Proof **(i)** Suppose $\frac{a^2}{4} < b \leq \frac{a^2}{3}$. Then $A(x_1) = 0$, and $A(x) > 0$ for $x \in [x_0, x_1)$. Since $B(x) > 0$ and $P(y) > 0$,

$$N(x, y) \geq 0 \iff \frac{A(x)}{B(x)} \geq \frac{Q(y)}{P(y)}.$$

Since Q/P is decreasing in y^2, $\frac{Q(y)}{P(y)} \leq \frac{Q(0)}{P(0)} = \frac{3c}{2}$. Moreover,

$$\frac{A(x_0)}{B(x_0)} \leq \frac{A(x)}{B(x)}, \qquad x \in [x_0, x_1].$$

Since

$$c \leq c_0 = \frac{2A(x_0)}{3B(x_0)},$$

we have

$$\frac{A(x)}{B(x)} \geq \frac{3c}{2} \geq \frac{Q(y)}{P(y)},$$

and therefore $N(x, y) \geq 0$ in $\Omega(E_0)$, so the conclusion follows from Theorem 1.1.

(ii) The proof follows the same argument as that of item (ii) in Lemma 3.9. □

Remark 3.12 In the cases where $\Delta < 0$, phenomena analogous to those described in Remark 3.7 may also occur. We illustrate this behavior with two representative examples.

First, consider the case $a = b = 3$, which falls under item (i) of Subcase (C3). A direct computation shows that $A(x)$ has a root at $x_1 = -\frac{1}{3}$. Since $F(x_0) = F(x_1) = \frac{1}{36}$, we obtain

$$x_0 \approx 0.1958, \qquad c_0 = \frac{2A(x_0)}{3B(x_0)} \approx 1.798.$$

Therefore, the period function is increasing on $\left(0, \frac{1}{36}\right)$. Numerical evidence suggests the existence of at least one critical point for $E > 1/36$. Moreover, if

$$c_0 < c < c_1 = \frac{2A(0)}{3B(0)} = 7,$$

then $T(E)$ remains increasing on $(0, F(x_c))$, where x_c is determined by $\sigma(x_c) = 0$. For $c > 7$, the period function becomes decreasing on $(0, F(x_c))$ and may exhibit critical points for $E > F(x_c)$. This behavior is observed numerically, for example, when $a = b = 3$ and $c = 8$.

Next, consider the case $a = b = 2$, which falls under item (ii) of Subcase (C3). In this case,

$$x_m \approx -0.0748706245, \qquad E_0 := F(x_m) \approx 0.0025387196, \qquad c_0 \approx 2.8004647.$$

Therefore, the period function is decreasing on $(0, E_0)$ for every $c \geq c_0$. Numerical evidence suggests that $T(E)$ may remain decreasing for $E > E_0$, as observed when $a = b = 2$ and $c = 3$.

On the other hand, for $0 < c < c_0$, the period function may be increasing for small values of E, as illustrated by the example $a = b = 2$ and $c = 1$.

The examples above indicate that the restriction to a subannulus is not merely a limitation of the criterion. Rather, the critical energy level reflects an actual change in the monotonicity of the period function. Consequently, one cannot, in general, expect the period function to be monotone throughout the entire period annulus.

It is worth emphasizing that the analysis presented for cases (A)–(C) does not cover all possible monotonicity behaviors of the period function $T(E)$. For instance, the case $c < 0$ has not been considered here, although it can be investigated using the same approach.

3.2 Relativistic Harmonic Oscillator

Example 3.13 *(Relativistic harmonic oscillator)* Consider the relativistic harmonic oscillator, whose Hamiltonian is given by

$$K(q, p) = \gamma mc^2 + \frac{1}{2}kq^2, \qquad \text{where} \quad \gamma = \sqrt{1 + \frac{p^2}{m^2c^2}}.$$

The first term represents the relativistic kinetic energy, while the second one corresponds to the potential energy of a linear restoring force.

Introducing the dimensionless variables

$$x = \sqrt{\frac{k}{mc^2}}\, q, \qquad y = \frac{p}{mc}, \qquad H = \frac{K}{mc^2},$$

the Hamiltonian takes the normalized form

$$H(x, y) = \frac{1}{2}x^2 + \sqrt{1 + y^2} - 1.$$

Applying Hamilton's equations, we obtain the system

$$x' = \frac{y}{\sqrt{1 + y^2}}, \qquad y' = -x,$$

which corresponds to the standard form of the relativistic harmonic oscillator (see [31]).

In that work, the authors showed, by directly analyzing the period function $T(E)$, that the oscillation period increases with the energy. We now recover this result as a direct consequence of Theorem 1.1.

Proposition 3.14 *Let $T(E)$ denote the period function associated with the center at $(0, 0)$. Then $T(E)$ is increasing on $(0, +\infty)$.*

Proof A direct computation shows that

$$M(x, y) = \frac{x^4}{2}\left((G')^2 - 2\,GG''\right).$$

Thus, it suffices to verify that $(G')^2 - 2\,GG'' \geq 0$ for all y.

Using $G(y) = \sqrt{1 + y^2} - 1$ and setting $s = \sqrt{1 + y^2} \geq 1$, we obtain

$$(G')^2 - 2\,GG'' = \frac{(s-1)^2(s+2)}{s^3}.$$

Since $(s-1)^2(s+2) \geq 0$ for all $s \geq 1$, with equality only at $s = 1$ (i.e., $y = 0$), it follows that $(G')^2 - 2\,GG'' \geq 0$ for all $y \in \mathbb{R}$. Hence, $M(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$.

By Theorem 1.1, the period function $T(E)$ is increasing on $(0, +\infty)$. □

3.2.1 Non-polynomial Systems

We now illustrate the applicability of Theorem 1.1 beyond the polynomial family considered above. The following examples show that the criterion can also be effectively applied to non-polynomial Hamiltonian systems, yielding explicit information on the monotonicity of the period function.

Example 3.15 *(A pair of pendulum–type equations coupled through their velocities)* Consider the planar system

$$x' = \sin y, \qquad y' = -\sin x,$$

which may be interpreted as a symmetric variant of the harmonic pendulum.

The system is Hamiltonian with the separable Hamiltonian

$$H(x, y) = F(x) + G(y), \qquad F(x) = 1 - \cos x, \qquad G(y) = 1 - \cos y.$$

The origin is a nondegenerate center, and its period annulus is bounded by the energy level $E_0 = 2$, corresponding to the separatrix connecting the saddle points $(0, \pm\pi)$ and $(\pm\pi, 0)$.

We analyze the monotonicity of the period function $T(E)$ for $E \in (0, 2)$.

A direct (though lengthy) computation shows that $M(x, y)$ can be written as

$$M(x, y) = 4\,(1 - \cos y)\,\sin^4\!\Big(\frac{x}{2}\Big)\,\sin^2\!\Big(\frac{y}{2}\Big)\,N(x, y),$$

where

$$N(x, y) = 5 + \cos(2x) - 2(\cos x - 2)\cos y.$$

Hence, the sign of $M(x, y)$ is determined by $N(x, y)$. A direct verification shows that $N(x, y) \geq 0$ on

$$\Omega(2) = \{(x, y) \in \mathbb{R}^2 : 0 \leq H(x, y) \leq 2\},$$

and therefore $M(x, y) \geq 0$ on $\Omega(2)$.

By Theorem 1.1, the period function $T(E)$ is increasing on $(0, 2)$.

Example 3.16 Consider the system

$$x' = \sinh(y), \qquad y' = -\sinh(x).$$

This system is Hamiltonian with

$$H(x, y) = \cosh(x) - 1 + \cosh(y) - 1,$$

and it possesses an unbounded global center at the origin.

We show that the period function $T(E)$ is decreasing on $(0,\ 4 + 4\sqrt{3}]$.

A direct computation yields

$$M(x, y) = -(\cosh x - 1)^2(\cosh y - 1)^2\big(\cosh^2 x - \cosh x\ \cosh y + 2\cosh y + 2\big).$$

Setting $a = \cosh x \geq 1$ and $b = \cosh y \geq 1$, we obtain

$$M(x, y) = -(a - 1)^2(b - 1)^2 f(a, b), \qquad f(a, b) = a^2 - ab + 2b + 2.$$

Thus, $M \leq 0$ if and only if $f \geq 0$. Moreover,

$$H(x, y) = a + b - 2.$$

Hence, determining the largest value E_0 for which $M \leq 0$ on $\Omega(E_0)$ reduces to minimizing $a + b$ subject to $f(a, b) = 0$. Solving $f(a, b) = 0$ for b gives

$$b(a) = \frac{a^2 + 2}{a - 2}, \qquad a > 2.$$

A direct computation shows that $S(a) = a + b(a)$ attains its minimum at

$$a_{\min} = 2 + \sqrt{3}.$$

Consequently,

$$E_0 = S(a_{\min}) - 2 = 4 + 4\sqrt{3}.$$

Therefore, $M(x, y) \leq 0$ for all (x, y) such that $0 < H(x, y) \leq E_0$, and by Theorem 1.1 the period function $T(E)$ is decreasing on $(0, E_0)$.

Theorem 1.1 ensures monotonicity up to the energy level E_0. For larger energies, the derivative $T'(E)$ may change sign, and no definitive conclusion can be drawn from the criterion alone.

The examples above illustrate the applicability of Theorem 1.1 to a variety of separable Hamiltonian systems, including non-polynomial models. They show that the proposed criterion can be effectively applied through explicit computations involving F, G, and their derivatives.

4 Comparison with the Criteria of Villadelprat and Zhang

In [29], the authors study the monotonicity of the period function associated with separable Hamiltonian systems of the form

$$H(x, y) = F_1(x) + F_2(y),$$

by means of the involutions associated with the functions F_i, $i = 1, 2$. More precisely, let $I_i = (e_i^-, e_i^+)$, $i = 1, 2$, denote the projections of the period annulus of the center $(0, 0)$ onto the coordinate axes. The involutions $\sigma_i : I_i \to I_i$ are defined implicitly by

$$F_i(\sigma_i(z)) = F_i(z), \qquad \forall\, z \in I_i.$$

The main monotonicity criteria obtained in [29] are formulated in terms of the operators

$$P_{\sigma_i}(f)(z) = \frac{f(z) - f(\sigma_i(z))\sigma_i'(z)}{2}.$$

For nondegenerate centers, the most relevant results for comparison with Theorem 1.1 are Proposition 3.2 and Corollary 3.5 of [29]. Proposition 3.2 states that if

$$P_{\sigma_i}\left(\left(\frac{F_i}{(F_i')^2}\right)''\right)$$

is positive (resp. negative) on $(0, e_i^+)$ for $i = 1, 2$, then the period function T is increasing (resp. decreasing) throughout the period annulus associated with the nondegenerate center $(0, 0)$.

The fact that this criterion depends explicitly on the involutions σ_i may hinder its practical applicability, since, in general, it is not possible to determine their algebraic expressions explicitly. This limitation is mentioned by the authors themselves, who also present particular situations in which this difficulty can be circumvented.

For nondegenerate centers, Corollary 3.5 provides a simpler version of the previous criterion. In this result, it suffices to analyze the sign of the expressions

$$\left(\frac{F_i}{(F_i')^2}\right)'', \qquad i = 1, 2. \tag{9}$$

More precisely, if the expressions in (9) are positive (resp. negative) on (e_i^-, e_i^+) for $i = 1, 2$, then the period function T is increasing (resp. decreasing) throughout the period annulus.

This criterion is particularly attractive from a computational point of view, since it reduces the study of the monotonicity of the period function to the independent analysis of two one-variable functions. However, it is only conclusive when both expressions in (9) have the same sign. In particular, the result provides no information about the monotonicity of T in cases where

$$\left(\frac{F_1}{(F_1')^2}\right)'' \quad \text{and} \quad \left(\frac{F_2}{(F_2')^2}\right)''$$

have opposite signs.

To illustrate this situation, we revisit the Hamiltonian considered in Example 3.8. As shown there, after the normalization (7), this system corresponds to Case (A) and exhibits a qualitative change in the behavior of the period function as the parameter k varies.

A direct computation yields

$$\left(\frac{F_1}{(F_1')^2}\right)'' = \frac{2x+5}{3(1+x)^4},$$

which is positive for all $x \in (-1, +\infty)$. On the other hand,

$$\left(\frac{F_2}{(F_2')^2}\right)'' = \frac{3k}{2}\frac{k^2y^4 + 4ky^2 - 1}{(1+ky^2)^4}.$$

Consequently,

$$\left(\frac{F_2}{(F_2')^2}\right)'' < 0 \quad \text{for} \quad 0 < |y| < \sqrt{\frac{\sqrt{5}-2}{k}}.$$

Therefore, the expressions in (9) have opposite signs in a neighborhood of the origin. Hence, neither Corollary 3.5 nor Proposition 3.2 allows one to conclude the monotonicity of the period function associated with the nondegenerate center $(0, 0)$.

By contrast, Example 3.8 shows that Theorem 1.1 guarantees that the period function $T(E)$ is increasing on $(0, 1/6)$ for every

$$0 < k \leq \frac{2}{9}.$$

Moreover, it provides the explicit threshold values

$$k_0 = \frac{2}{9}, \qquad k_1 = \frac{10}{9},$$

which explain the qualitative transition observed numerically in Fig. 2 of [14]. Therefore, Theorem 1.1 continues to provide rigorous analytical information even when the criteria in [29] are inconclusive.

On the other hand, there are situations in which the criterion of Villadelprat and Zhang yields conclusions more quickly than Theorem 1.1. This can be illustrated by the system

$$x' = \sin y, \qquad y' = -\sin x,$$

discussed in Sect. 3. In this case,

$$F_1(z) = F_2(z) = 1 - \cos z,$$

and a direct computation shows that

$$\left(\frac{F_i}{(F_i')^2}\right)''(z) = \frac{1}{4}(2 - \cos z)\sec^4\left(\frac{z}{2}\right), \qquad i = 1, 2.$$

Since this expression is positive on $(-\pi, \pi)$, Corollary 3.5 immediately guarantees that the period function is increasing throughout the period annulus associated with the center. In contrast, the application of Theorem 1.1 requires an additional analysis of the sign of the function $M(x, y)$.

The examples above suggest the following practical strategy. Given a separable Hamiltonian system, one should first verify the hypotheses of Corollary 3.5 by analyzing the sign of the functions

$$\left(\frac{F_i}{(F_i')^2}\right)'', \qquad i = 1, 2.$$

Whenever these hypotheses are satisfied, the monotonicity of the period function follows immediately. Otherwise, one may resort to Theorem 1.1, which, as illustrated

by the previous example, may provide information on the monotonicity of the period function even when the hypotheses of the previous criteria are not satisfied. The two approaches should therefore be regarded as complementary rather than competing.

5 Conclusion

In this paper, we investigated the monotonicity of the period function associated with planar Hamiltonian systems of the form

$$H(x, y) = F(x) + G(y),$$

where G is an even function. We established explicit sufficient conditions for monotonicity in terms of the functions F and G through the function $M(x, y)$.

Our main result extends Chicone's classical criterion from the potential case to a class of separable Hamiltonian systems. Moreover, it retains the direct formulation of Chicone's criterion in terms of F, G, and their derivatives, while extending its applicability to a broader class of separable Hamiltonian systems.

To illustrate the applicability of the proposed criterion, we considered polynomial, trigonometric, hyperbolic, and relativistic Hamiltonian systems. In addition, a comparison with a previously established criterion shows that the proposed criterion can establish monotonicity in situations where the previous criterion is not conclusive.

Finally, the approach developed in this paper may also be useful in the study of other classes of Hamiltonian systems and related questions concerning the monotonicity of the period function.

Appendix A Proof of the Positivity of $A(x)$

This appendix completes the proof of Subcase (B3) by establishing the positivity of the polynomial A on the projection of the period annulus. More precisely, we prove that the polynomial

$$A(x) = 10a^2 - 9b + a(30b + 4a^2)x + b(36b + 16a^2)x^2 + 24ab^2x^3 + 9b^3x^4$$

is positive on the interval $[x_0, r_0]$, where $[x_0, r_0]$ denotes the projection of the period annulus associated with the center at the origin.

Throughout this appendix, we assume that $a > 0$ and $a^2 - 4b > 0$. Let $x_0 < 0$ be the negative root of

$$p(x) = 1 + ax + bx^2,$$

and let $r_0 > 0$ be the unique solution of

$$F(r_0) = F(x_0), \qquad F(x) = \frac{x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}.$$

The main result of this appendix is the following.

Lemma A.1 *The polynomial A satisfies*

$$A(x) > 0, \qquad x \in [x_0, r_0].$$

The proof is divided according to the sign of b. The case $0 < b < a^2/4$ is treated by elementary arguments, whereas the case $b < 0$ requires Sturm's theorem.

A.1 The Case $0 < b < a^2/4$

Since $A(-x, -a, b) = A(x, a, b)$ and $F(-x, -a, b) = F(x, a, b)$, it is enough to assume $a > 0$. As all coefficients of A are positive, we have $A(x) > 0$ for $x \geq 0$. Thus, it remains to prove the positivity of A on $[x_0, 0]$. Using the identity

$$A(x) = p(x)q(x) + 3(a^2 - 4b)(ax + 3),$$

where

$$p(x) = 1 + ax + bx^2, \qquad q(x) = a^2 + 27b + 15abx + 9b^2x^2,$$

it suffices to prove that $q(x) > 0$ and $ax + 3 > 0$ on $[x_0, 0]$, since $p(x) \geq 0$ on this interval and $a^2 - 4b > 0$.

First, $p(-2/a) = (4b - a^2)/a^2 < 0$ and $p(0) = 1 > 0$, so $-\frac{2}{a} < x_0 < 0$. Hence,

$$ax + 3 > 1 > 0, \qquad x \in [x_0, 0].$$

Moreover,

$$q'(x) = 3b(5a + 6bx) > \frac{3b(5a^2 - 12b)}{a} > 0, \qquad x \in [x_0, 0],$$

because $0 < b < a^2/4$. Therefore, q is increasing on $[x_0, 0]$, and $q(x) \geq q(x_0)$. Using $1 + ax_0 + bx_0^2 = 0$,

$$q(x_0) = a^2 + 6b(ax_0 + 3) > a^2 > 0.$$

Hence $q(x) > 0$ on $[x_0, 0]$, and consequently

$$A(x) > 0, \qquad x \in [x_0, 0].$$

Since $A(x) > 0$ for $x \geq 0$, we conclude that

$$A(x) > 0, \qquad x \in [x_0, r_0].$$

A.2 The Case $b < 0$

When $b < 0$, the previous argument no longer applies, since the polynomial

$$q(x) = a^2 + 27b + 15abx + 9b^2x^2$$

is not necessarily positive on $[x_0, r_0]$. Consequently, the identity

$$A(x) = p(x)q(x) + 3(a^2 - 4b)(ax + 3)$$

does not directly imply the positivity of A.

Instead, we apply Sturm's theorem to show that A has no real zeros on $[x_0, r_0]$. Since $A(0) = 10a^2 - 9b > 0$, it follows that $A(x) > 0$ for every $x \in [x_0, r_0]$. The proof is divided into four steps.

A.2.1 The Sturm Sequence

The Sturm sequence associated with A begins with $p_0 = A$, $p_1 = A'$, and the remaining polynomials are

$$p_2(x) = \frac{(2a^2 - 9b)(2a^2 + 7abx + 12b^2x^2 - 6b)}{6b},$$

$$p_3(x) = \frac{3}{4}\left(6a^3 + 5a^2bx - 74ab - 120b^2x\right),$$

$$p_4 = -\frac{4(a^2 - 4b)(2a^2 - 9b)(34a^4 - 561a^2b + 2700b^2)}{75b(a^2 - 24b)^2}.$$

Since p_4 is a nonzero constant, the Euclidean algorithm terminates and $\{p_0, p_1, p_2, p_3, p_4\}$ is the Sturm sequence associated with A.

A.2.2 The Sign of p_2

Since

$$p_2(x) = \frac{(2a^2 - 9b)(2a^2 + 7abx + 12b^2x^2 - 6b)}{6b},$$

and the quadratic factor has discriminant $\Delta = b^2(288b - 47a^2) < 0$ and positive leading coefficient $12b^2$, it follows that this factor is positive on $\mathbb{R}$. Therefore,

$$p_2(x) < 0, \qquad x \in \mathbb{R},$$

since $2a^2 - 9b > 0$ and $6b < 0$.

A.2.3 The Sign of p_3

Since

$$p_3(x) = \frac{3}{4}\left(6a^3 + 5a^2bx - 74ab - 120b^2x\right),$$

it is an affine polynomial whose unique zero is

$$x_* = \frac{2a(37b - 3a^2)}{5b(a^2 - 24b)}.$$

To compare x_* with r_0, we eliminate x_0 from the system

$$p(x_0) = 0, \qquad F(r_0) = F(x_0),$$

obtaining

$$(ar_0 + br_0^2 + 1)^2\, G(r_0) = 0,$$

where

$$L(x) = 4a^3x - 5a^2bx^2 - 2a^2 - 18abx + 18b^2x^2 + 6ab^2x^3 + 9b^3x^4 + 9b.$$

Since $p(r_0) > 0$, we have $L(r_0) = 0$. A symbolic computation yields

$$\mathrm{Res}(p_3, L) = -b^3(a^2 - 4b)(-b)^5 P\left(\frac{a^2}{-b}\right),$$

where

$$P(u) = 2316u^5 + 419042u^4 + 15032343u^3 + 199100124u^2 + 865468800u - 466560000.$$

A direct computation shows that $P'(u) > 0$, $u > 0$. Since $P(4) > 0$, we conclude that $P(u) > 0$, $u > 4$. Since $u = \frac{a^2}{-b} > 4$, we obtain

$$\mathrm{Res}(p_3, L) > 0,$$

and hence $p_3(r_0) \neq 0$. Since $F'(r_0) = r_0 p(r_0) > 0$, the Implicit Function Theorem implies that r_0 depends continuously on (a, b). As the parameter region

$$\{(a, b) : a > 0,\ b < 0,\ a^2 - 4b > 0\}$$

is connected, the sign of $p_3(r_0)$ is constant. Evaluating at $(a, b) = \left(1, -\frac{1}{10}\right)$ gives $p_3(r_0) > 0$. Consequently, $p_3(r_0) > 0$ for every admissible pair. Finally,

$$p_3'(x) = \frac{15}{4}b(a^2 - 24b) < 0,$$

so p_3 is strictly decreasing. Therefore, $x_* > r_0$, and

$$p_3(x) > 0, \qquad x \in [x_0, r_0].$$

A.2.4 The Sign of p_4

The constant polynomial p_4 is given by

$$p_4 = -\frac{4(a^2-4b)(2a^2-9b)(34a^4-561a^2b+2700b^2)}{75b(a^2-24b)^2}.$$

Since $a^2-4b > 0$, $2a^2-9b > 0$, and $75b(a^2-24b)^2 < 0$, it remains to determine the sign of

$$34a^4 - 561a^2b + 2700b^2.$$

Writing $u = a^2/(-b) > 4$, we obtain

$$34a^4 - 561a^2b + 2700b^2 = b^2(34u^2 + 561u + 2700) > 0,$$

since $b^2 > 0$. Therefore,

$$p_4 > 0.$$

A.2.5 Positivity of A

We now prove that

$$A(x) > 0, \qquad x \in [x_0, r_0].$$

From the previous subsections,

$$p_2(x) < 0, \qquad p_3(x) > 0, \qquad p_4 > 0,$$

for every $x \in [x_0, r_0]$. Moreover,

$$A(x_0) = 3(a^2-4b)(ax_0+3) > 0,$$

and

$$A(r_0) = p(r_0)q(r_0) + 3(a^2-4b)(ar_0+3) > 0.$$

Thus, the Sturm sequence has signs

$$(+, \pm, -, +, +)$$

at both endpoints, so

$$V(x_0) = V(r_0) = 2.$$

By Sturm's theorem, A has no real zeros in $[x_0, r_0]$. Since

$$A(0) = 10a^2 - 9b > 0,$$

we conclude that

$$A(x) > 0, \qquad x \in [x_0, r_0].$$

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