Qualitative Theory of Dynamical Systems (2026) 25:48
https://doi.org/10.1007/s12346-026-01469-1

Mechanical Normal Forms for Analytic Centers

Francisco J. S. Nascimento[1]

Received: 22 November 2025 / Accepted: 2 February 2026 / Published online: 28 February 2026
© The Author(s) 2026

Abstract
The classical Poincaré Normal Form Theorem asserts that a singular point of an analytic planar vector field is a non-degenerate center if and only if, after an analytic change of coordinates, the system can be written in the rotational normal form

$$f(x^2 + y^2)\big(y\,\partial_x - x\,\partial_y\big), \qquad f(0) > 0.$$

In this paper we prove that every analytic planar vector field with a non-degenerate center at the origin is locally analytically conjugate to a one-degree-of-freedom mechanical Hamiltonian system

$$y\,\partial_x - V'(x)\,\partial_y,$$

where V is analytic and satisfies $V(0) = V'(0) = 0$ and $V''(0) > 0$. The construction of V is completely explicit and depends solely on the period function of the original center. Consequently, the local analytic classification of non-degenerate centers reduces to the classification of analytic potentials, or equivalently, of their period functions. Our result provides a local analytic answer to a question related to Chicone's 1987 work, where he established a celebrated criterion for studying the monotonicity of the period function of mechanical Hamiltonian systems using only the potential V and its derivatives V', V'', and V'''. In this sense, our theorem shows that the local monotonicity problem for the period function of an arbitrary analytic vector field with a non-degenerate center reduces to the monotonicity problem for the period function of an associated mechanical system.

Keywords Non-degenerate center · Period function · Analytic conjugacy · Mechanical Hamiltonian system

✉ Francisco J. S. Nascimento
francisco.jsn@univasf.edu.br

[1] Colegiado de Geologia, Universidade Federal do Vale do São Francisco, Senhor do Bonfim, BA, Brazil

1 Introduction and Main Results

Let Ω be an open subset of $\mathbb{R}^2$, and let $X = (P, Q) \in C^\omega(\Omega, \mathbb{R}^2)$ be the analytic planar vector field defined by

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \qquad (x, y) \in \Omega. \tag{1}$$

Throughout the paper, we also represent X by the differential operator

$$X = P\,\partial_x + Q\,\partial_y. \tag{2}$$

A point $p \in \Omega$ such that $X(p) = (0, 0)$ is called a *singular point* of X. The singular point p is said to be *nondegenerate* if the Jacobian matrix $DX(p)$ has nonzero determinant, that is,

$$\det DX(p) = P_x(p)Q_y(p) - P_y(p)Q_x(p) \neq 0.$$

A singular point p is a *center* of X if there exists an open neighborhood U of p such that every solution of (1) with initial condition in $U \setminus \{p\}$ is a periodic orbit surrounding p. The largest such neighborhood will be denoted by $\mathcal{A}$ and is called the *period annulus* of the center.

If p is a center and $q \in \mathcal{A}$, we denote by $T(q)$ the period of the periodic orbit passing through q. The map

$$T : \mathcal{A} \to \mathbb{R}_{>0}, \qquad q \mapsto T(q),$$

is called the *period function* of the center p.

Let Ω_1, Ω_2 be open subsets of $\mathbb{R}^2$. Two analytic vector fields $X_1 \in C^\omega(\Omega_1, \mathbb{R}^2)$ and $X_2 \in C^\omega(\Omega_2, \mathbb{R}^2)$ are said to be *analytically equivalent*, or *analytically conjugate*, if there exists an analytic diffeomorphism $h : \Omega_1 \to \Omega_2$ such that

$$dh(z)\, X_1(z) = X_2\big(h(z)\big), \qquad \text{for all } z \in \Omega_1. \tag{3}$$

Equivalently,

$$X_2(w) = dh\big(h^{-1}(w)\big)\, X_1\big(h^{-1}(w)\big) =: (h_* X_1)(w), \qquad \text{for all } w \in \Omega_2. \tag{4}$$

Such a map h sends singular points to singular points, periodic orbits to periodic orbits, and preserves the periods of corresponding periodic orbits. Throughout the paper, such a diffeomorphism h will frequently be referred to as a *change of coordinates*.

If p is a singular point of X_1, we say that X_1 is *locally analytically conjugate* to X_2 at p if (3) holds on some neighborhood of p. Throughout this work, we assume without loss of generality that the singular point is the origin $p = (0, 0)$. Accordingly, we write $h : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ to indicate that h is a local analytic diffeomorphism defined in a neighborhood of the origin and satisfying $h(0, 0) = (0, 0)$.

The classical Poincaré Normal Form Theorem (see, e.g., [1, 2]) states that the origin is a non-degenerate center of an analytic planar vector field X if and only if there exists a local analytic diffeomorphism $h : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$, such that the vector field is transformed into the rotational normal form

$$h_* X = X_f = f(x^2 + y^2)\big(y\,\partial_x - x\,\partial_y\big),$$

where f is analytic in a neighborhood of the origin and satisfies $f(0) > 0$.

In particular, the center is *isochronous* (i.e. its period function is constant) if and only if there exists a local analytic diffeomorphism $h : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ such that

$$h_* X = X_\omega = \omega\big(y\,\partial_x - x\,\partial_y\big), \qquad \omega > 0,$$

see [3, Theorem 3.3].

When the period function is not constant, that is, when $T'(E) \neq 0$, one may ask whether it is still possible to simplify the vector field X_f by an analytic change of coordinates. Our main result, stated below, shows that the answer is yes.

Theorem 1.1 *Let X be a real-analytic planar vector field having a non-degenerate center at the origin* $(0, 0)$. *Then there exist an* even *real-analytic function V, defined in a neighborhood of* 0 *and satisfying*

$$V(0) = V'(0) = 0, \qquad V''(0) > 0,$$

and a local analytic diffeomorphism $h : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ such that $h_* X = X_V$, where X_V is the mechanical Hamiltonian vector field

$$X_V = y\,\partial_x - V'(x)\,\partial_y. \tag{5}$$

A global counterpart of Theorem 1.1 was established by the author in collaboration with C. G. Ragazzo in [4, Theorem 1.1, Theorem 1.2]. Theorem 1.1 of [4] shows that if a real-analytic vector field $X(u, v)$ has a non-degenerate center at the origin, regardless of whether the center is global, then for every disk D_γ bounded by a periodic orbit γ in the period annulus, there exists an analytic change of variables $\Lambda(u, v) = (x, y)$, defined on the whole disk D_γ, which transforms X into the mechanical vector field $X_V(x, y) = y\,\partial_x - V'(x)\,\partial_y$.

In this sense, the result of [4] is genuinely global, whereas Theorem 1.1 of the present paper is strictly local, holding only in a sufficiently small neighborhood of the origin, even when the center is global.

It is also important to emphasize that the methods used to prove these two results are completely different. The proof of the local case relies on the analytic Poincaré normal form together with the construction of a potential V obtained directly from the local period function. In contrast, the proof in [4] employs entirely different techniques, including a global version of Poincaré's theorem and the Abel integral equation to

construct a global potential V. The analytic conjugacy $\Lambda\colon (u, v) \mapsto (x, y)$ is constructed using the flows of the vector fields X and X_V, and does not use power-series normal forms at any step.

Thus, the present article provides the *local* counterpart of the global result in [4], and the methods used in the two works are independent and mutually complementary. In particular, the main contribution of the present work does not lie merely in the statement of the local normalization result, but rather in the constructive methods employed in its proof, most notably in the Fundamental Lemma 2.3, which provides an explicit analytic construction of the mechanical potential directly from the local period function.

The vector field in (5) corresponds to the mechanical system

$$x' = y, \qquad y' = -V'(x),$$

usually called a potential system. Such systems have been extensively studied in the literature, and two classical questions about them are directly related to the period function associated with their non-degenerate centers.

The first question concerns the *monotonicity* of the period function, a topic addressed in several works, including [3, 5].

The second is the so-called *inverse problem*, which asks whether a given positive real-analytic function T can be realized as the period function of some analytic potential V. This problem has been studied in [6–8].

In [5], Chicone also asked under what conditions a planar vector field with a center at the origin is analytically conjugate to a mechanical Hamiltonian system of this type. Theorem 1.1 provides a *local analytic* answer to this question in the case where the vector field is real-analytic and the center is non-degenerate.

The proof of Theorem 1.1 is entirely constructive, and the construction of the potential V depends solely on the period function associated with the original center. In this sense, Theorem 1.1 shows that, up to local analytic conjugacy, every non-degenerate center is mechanically realizable, that is, it can be represented as the motion of a unit mass in a one-dimensional analytic potential V. Consequently, the local analytic classification of non-degenerate centers reduces to the study of analytic potentials, or, equivalently, of their period functions.

2 Proof of Theorem 1.1

Lemma 2.1 *Let X be a real-analytic planar vector field having a non-degenerate center at the origin $(0, 0)$. Then there exists a local real-analytic diffeomorphism $h_1 : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$, such that*

$$(h_1)_* X = f(\xi^2 + \eta^2)\big(\eta\, \partial_\xi - \xi\, \partial_\eta\big), \tag{6}$$

where f is a real-analytic function defined in a neighborhood of the origin and satisfies $f(0) > 0$.

Proof This is a direct consequence of the Poincaré–Lyapunov analytic linearization theorem, see [2, Theorem 3.1]. □

The vector field (6) is Hamiltonian with Hamiltonian function

$$H_f(\xi, \eta) = F(\xi^2 + \eta^2), \qquad F(z) = \frac{1}{2}\int_0^z f(s)\, ds.$$

In particular, the Hamiltonian H_f provides a natural parametrization of the periodic orbits of (6) by their energy levels. More precisely, for each energy value $E > 0$ sufficiently small, there exists a unique periodic orbit of X_f contained in the level set $\{H_f = E\}$.

We denote by $T(E; X_f, H_f)$ the period of this periodic orbit. Thus, the map

$$E \longmapsto T_f(E) := T(E; X_f, H_f)$$

defines the *period function* of the vector field X_f, expressed in terms of the energy parameter E.

Lemma 2.2 *Let $T_f(E) = T(E; X_f, H_f)$ denote the period function of the vector field X_f associated with the Hamiltonian H_f. Then $T_f(E)$ is real-analytic near $E = 0$ and satisfies*

$$T_f(E) = \frac{\pi}{F'\big(F^{-1}(E)\big)} = \pi\, \frac{d}{dE} F^{-1}(E). \tag{7}$$

Proof In polar coordinates $\xi = r\cos\theta$, $\eta = r\sin\theta$, the vector field (6) takes the form

$$\dot r = 0, \qquad \dot\theta = f(r^2). \tag{8}$$

Thus, every periodic orbit corresponds to a circle $r = \text{const}$, and its period is given by

$$T(r) = \frac{2\pi}{f(r^2)}. \tag{9}$$

Since the Hamiltonian associated with (6) is

$$H_f(\xi, \eta) = F(r^2),$$

the energy level $H_f = E$ corresponds to $r^2 = F^{-1}(E)$. Substituting this relation into (9), we obtain

$$T_f(E) = T\big(F^{-1}(E)\big) = \frac{2\pi}{f\big(F^{-1}(E)\big)} = \frac{\pi}{F'\big(F^{-1}(E)\big)} = \pi\, \frac{d}{dE} F^{-1}(E).$$

Since $f(0) = F'(0) > 0$, the inverse function F^{-1} is real-analytic in a neighborhood of $E = 0$, and therefore the period function $T_f(E)$ is real-analytic near $E = 0$, with $T_f(0) = \frac{\pi}{F'(0)}$. □

Lemma 2.3 (Fundamental Lemma) *Let $T_f(E) = T(E; X_f, H_f)$ be the real-analytic period function defined in (7). Then there exists a real-analytic* even *function V, defined in a neighborhood of* 0*, such that*

$$V(0) = V'(0) = 0, \qquad V''(0) > 0.$$

Moreover, the Hamiltonian vector field

$$X_V(x, y) = y\,\partial_x - V'(x)\,\partial_y, \tag{10}$$

with Hamiltonian function

$$H_V(x, y) = \tfrac{1}{2}y^2 + V(x),$$

has period function $T_V(E) := T(E; X_V, H_V)$ satisfying

$$T_V(E) = T_f(E).$$

Proof The proof is divided into two steps. First, we construct a real-analytic even potential V satisfying the stated properties. In the second step, we show that the period function $T_V(E) = T(E; X_V, H_V)$ associated with the Hamiltonian vector field X_V defined by V coincides with the function $T_f(E)$.

Since $F^{-1}(E)$ is real-analytic in a neighborhood of $E = 0$ and satisfies $F^{-1}(0) = 0$, there exists a sequence of real numbers $(a_n)_{n\geq 1}$ such that

$$F^{-1}(E) = \sum_{n\geq 1} a_n E^n.$$

Differentiating term by term yields

$$\left(F^{-1}\right)'(E) = \sum_{n\geq 1} n a_n E^{n-1}. \tag{11}$$

Since $\left(F^{-1}\right)'(0) = 1/F'(0)$, we have $a_1 = 1/F'(0) > 0$.

Define a sequence $(b_n)_{n\geq 1}$ by

$$b_n = \frac{n\sqrt{2\pi}\,\Gamma(n)}{4\,\Gamma(n+\frac{1}{2})}\,a_n,$$

where Γ denotes the Euler gamma function. As

$$\lim_{n\to\infty} \frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} = 0,$$

there exists a constant $M > 0$ such that $|b_n| \leq nM|a_n|$ for all n. Consequently, the function

$$\varphi(z) = \sum_{n\geq 1} b_n z^{2n-1}$$

is real-analytic in a neighborhood of 0, with

$$\varphi(0) = 0, \qquad \varphi'(0) = b_1 = \frac{\sqrt{2}}{2} a_1 > 0.$$

Define

$$\zeta(E) = \varphi(\sqrt{E}) = \sum_{n\geq 1} b_n E^{(2n-1)/2}, \tag{12}$$

for E in a neighborhood of 0^+. By the chain rule,

$$\zeta'(E) = \frac{\varphi'(\sqrt{E})}{2\sqrt{E}} = \frac{b_1}{2\sqrt{E}} + \sqrt{E}\,(b_2 + \cdots),$$

and therefore $\zeta'(E) \to +\infty$ as $E \to 0^+$. In particular, ζ is strictly increasing and hence locally invertible near $E = 0$.

Since $\zeta(E) = \varphi(\sqrt{E})$, the inverse function ζ^{-1} of $x = \zeta(E)$ satisfies

$$\zeta^{-1}(x) = \left(\varphi^{-1}(x)\right)^2.$$

We define the potential V in a neighborhood of 0 by

$$V(x) = \zeta^{-1}(x) = \left(\varphi^{-1}(x)\right)^2.$$

By construction, V is real-analytic near 0 and satisfies

$$V(0) = V'(0) = 0, \qquad V''(0) = 2(\varphi^{-1})'(0) = \frac{2}{\varphi'(0)} = \frac{2}{b_1} > 0.$$

Moreover, since φ is an odd analytic function, its inverse φ^{-1} is also odd, and therefore V is an even function.

This completes the proof of the existence of the potential V with the stated properties.

We now consider the Hamiltonian vector field X_V defined by

$$X_V(x, y) = y\,\frac{\partial}{\partial x} - V'(x)\,\frac{\partial}{\partial y}. \tag{13}$$

By construction, X_V is analytic in a neighborhood of $(0, 0)$ and has a non-degenerate center at the origin. In particular, all periodic orbits of (13) are contained in the level

sets $\{H_V(x, y) = E\}$, where H_V is the associated Hamiltonian function, defined in a neighborhood of $(0, 0)$ by

$$H_V(x, y) = \frac{y^2}{2} + V(x). \tag{14}$$

We show next that the period function T_V associated with the vector field X_V coincides with the function T_f.

By definition, the period $T_V(E) = T(E; X_V, H_V)$ of the periodic orbit of X_V contained in the energy level $H_V = E$ is given by

$$\begin{aligned} T_V(E) &= 2\int_{V_-^{-1}(E)}^{V_+^{-1}(E)} \frac{dx}{y} = 2\int_{V_-^{-1}(E)}^{V_+^{-1}(E)} \frac{dx}{\sqrt{2(E - V(x))}} \\ &= \sqrt{2}\int_0^{V_+^{-1}(E)} \frac{dx}{\sqrt{E - V(x)}} - \sqrt{2}\int_0^{V_-^{-1}(E)} \frac{dx}{\sqrt{E - V(x)}}, \end{aligned}$$

where V_-^{-1} and V_+^{-1} denote, respectively, the inverse branches of V defined on $x < 0$ and $x > 0$.

Performing the changes of variables $x = V_+^{-1}(s)$ and $x = V_-^{-1}(s)$ in the two integrals above, we obtain

$$T_V(E) = \sqrt{2}\int_0^E \frac{\left(V_+^{-1}(s) - V_-^{-1}(s)\right)'}{\sqrt{E - s}}\, ds. \tag{15}$$

Since V is an even function, we have $V_-^{-1}(E) = -V_+^{-1}(E)$ and therefore

$$V_+^{-1}(E) - V_-^{-1}(E) = 2V_+^{-1}(E) = 2V^{-1}(E).$$

Consequently, the period function $T_V(E)$, parametrized by the energy levels $H_V = E$, is given by the analytic expression

$$T_V(E) = 2\sqrt{2}\int_0^E \frac{(V^{-1})'(s)}{\sqrt{E - s}}\, ds. \tag{16}$$

By definition,

$$V^{-1}(E) = \varphi(\sqrt{E}) = \sum_{n\geq 1} b_n E^{(2n-1)/2},$$

and therefore

$$(V^{-1})'(E) = \sum_{n\geq 1} \frac{2n-1}{2}\, b_n E^{(2n-3)/2}.$$

Substituting this series into the expression for $T_V(E)$ and performing the change of variables $s = Et$, we obtain

$$T_V(E) = 2\sqrt{2}\int_0^1 \frac{\sum_{n\geq 1} \frac{2n-1}{2} b_n E^{n-1} t^{(2n-3)/2}}{\sqrt{1-t}}\, dt.$$

Define, for $t \in (0, 1)$ and $E > 0$ sufficiently small,

$$f_n(t, E) = \frac{2n-1}{2}\, b_n\, E^{n-1}\, \frac{t^{(2n-3)/2}}{\sqrt{1-t}}, \qquad n \geq 1.$$

For each fixed E, the series $\sum_{n\geq 1} f_n(t, E)$ converges pointwise on $(0, 1)$. Moreover, since $0 \leq t^{(2n-3)/2} \leq 1$ for $t \in [0, 1]$, there exists a constant $C(E) > 0$ such that

$$|f_n(t, E)| \leq C(E)\, n|b_n|\, E^{n-1}\, \frac{1}{\sqrt{1-t}}.$$

As $\sum_{n\geq 1} n|b_n|E^{n-1} < \infty$ for E sufficiently small and $(1-t)^{-1/2} \in L^1(0, 1)$, the series is dominated by an integrable function independent of n. Hence, by the Dominated Convergence Theorem, we may interchange summation and integration.

Consequently,

$$T_V(E) = 2\sqrt{2}\sum_{n\geq 1} \frac{2n-1}{2}\, b_n E^{n-1} \int_0^1 \frac{t^{(2n-3)/2}}{\sqrt{1-t}}\, dt.$$

Using the identity

$$\int_0^1 \frac{t^{(2n-3)/2}}{\sqrt{1-t}}\, dt = \frac{\sqrt{\pi}\,\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)},$$

together with the functional relation of the Gamma function

$$\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\Gamma\left(n-\tfrac{1}{2}\right),$$

we obtain

$$T_V(E) = 2\sqrt{2}\sum_{n\geq 1} b_n \frac{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}{\Gamma(n)} E^{n-1}.$$

By definition,

$$b_n = \frac{n\sqrt{2\pi}\,\Gamma(n)}{4\,\Gamma\left(n+\frac{1}{2}\right)}\, a_n.$$

Hence,

$$b_n \frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)} E^{n-1} = \frac{n\sqrt{2\pi}\,\Gamma(n)}{4\,\Gamma\left(n+\frac{1}{2}\right)}\, a_n \cdot \frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)} = \frac{\sqrt{2}\pi}{4}\, n a_n\, E^{n-1}.$$

Substituting this into the previous expression for $T_V(E)$, we obtain

$$T_V(E) = 2\sqrt{2}\sum_{n\geq 1}\frac{\sqrt{2}\pi}{4}\, n a_n\, E^{n-1} = \pi\sum_{n\geq 1} n a_n E^{n-1}.$$

By (11), this yields

$$T_V(E) = \pi\,(F^{-1})'(E) = \frac{\pi}{F'\big(F^{-1}(E)\big)} = T_f(E).$$

Hence the period functions coincide for E sufficiently close to 0, which completes the proof.

□

Remark 2.4 The constructive proof of the Fundamental Lemma 2.3 yields an *even* potential V. This symmetry plays a crucial technical role in the computation of the period function, since it allows one to identify the inverse branches of V by

$$V_-^{-1}(E) = -V_+^{-1}(E),$$

leading to a simplified integral expression for $T_V(E)$. It is precisely this simplified form that makes it possible to establish the equality of the period functions in the proof of Theorem 1.1.

Once the rotational normal form X_f of the vector field X is fixed, the potential V obtained by this construction is unique. In particular, the parity of V does not depend on the choice of the function f.

We also note that there are results in the literature ensuring the *local* existence and uniqueness of an even real-analytic potential V associated with a given positive real-analytic period function $T(E)$. In particular, in [7], Hamiltonian systems of the form

$$H(x,y) = H_0(x^2,y^2) + V(x^2), \quad H_0(x^2,y^2) = y^2 + \mathcal{O}(x^4,y^4), \quad V(x^2) = x^2 + \mathcal{O}(x^4),$$

are considered, and it is shown that, for a given function $T(E)$, there exists a unique even potential V such that the period function associated with the Hamiltonian system X_H coincides with $T(E)$. The classical mechanical potential system associated with the potential obtained in Lemma 2.3 corresponds to the particular case

$$H(x,y) = H_0(0, y^2/2) + V(x^2).$$

Although this result guarantees the existence and uniqueness of the potential V in Lemma 2.3, it is not sufficient for the purposes of the present work. Indeed, such abstract existence results do not provide a transparent relation between the coefficients of the normal form vector field X_f and those of the corresponding potential V. In contrast, our construction yields a potential V that depends explicitly on the coefficients of X_f, and this explicit dependence is essential for the subsequent steps in the proof of Theorem 1.1.

The next and final lemma provides an important canonical normalization result for analytic Hamiltonian centers.

Lemma 2.5 *Let H be a real-analytic function defined in a neighborhood of the origin and satisfying $H(0,0) = 0$. Assume that the Hamiltonian vector field*

$$X_H = H_y\,\partial_x - H_x\,\partial_y$$

has a non-degenerate center at the origin. Then there exists a local analytic diffeomorphism

$$\phi : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0), \qquad \det D\phi \equiv 1,$$

such that

$$\phi_* X_H = X_g = g(\xi^2 + \eta^2)\big(\eta\,\partial_\xi - \xi\,\partial_\eta\big),$$

where g is a real-analytic function defined near the origin and satisfies $g(0) > 0$. Moreover, if

$$H_g(\xi, \eta) = G(\xi^2 + \eta^2), \qquad G(z) = \frac{1}{2}\int_0^z g(s)\,ds,$$

denotes the Hamiltonian function associated with X_g, then the Hamiltonians satisfy

$$H = H_g \circ \phi.$$

Proof The result follows from the canonical normalization theory for analytic Hamiltonian centers developed by Mañosas and Villadelprat (see [9, Theorem A]). The existence of a real-analytic canonical change of coordinates ϕ satisfying the stated conditions is guaranteed by that theorem.

Moreover, since

$$\phi_* X_H = X_g,$$

a direct computation yields

$$\begin{pmatrix} H_y \\ -H_x \end{pmatrix} = \frac{1}{\det D\phi}\begin{pmatrix} (H_g \circ \phi)_y \\ -(H_g \circ \phi)_x \end{pmatrix}.$$

As ϕ is canonical, that is, $\det D\phi \equiv 1$, we obtain

$$\nabla H = \nabla(H_g \circ \phi),$$

and therefore there exists a constant $c \in \mathbb{R}$ such that

$$H = H_g \circ \phi + c.$$

Since $H(0,0) = H_g(0,0) = 0$ and $\phi(0,0) = (0,0)$, it follows that $c = 0$, and hence

$$H = H_g \circ \phi.$$

□

Conclusion of the Proof of Theorem 1.1

By Lemma 2.1, there exists a local analytic diffeomorphism h_1 such that

$$(h_1)_* X = X_f = f(\xi^2 + \eta^2)\big(\eta\,\partial_\xi - \xi\,\partial_\eta\big).$$

Let $T_f(E) = T(E; X_f, H_f)$ denote the period function of the vector field X_f obtained in Lemma 2.2. By Lemma 2.3, there exists an even real-analytic potential V such that the associated mechanical Hamiltonian system

$$X_V = y\,\partial_x - V'(x)\,\partial_y, \qquad H_V(x, y) = \frac{y^2}{2} + V(x),$$

has the same period function, namely,

$$T_V(E) = T(E; X_V, H_V) = T_f(E).$$

By Lemma 2.5, there exists a local analytic canonical diffeomorphism h_2, satisfying $\det Dh_2 \equiv 1$, such that

$$(h_2)_* X_V = X_g = g(\xi^2 + \eta^2)\big(\eta\,\partial_\xi - \xi\,\partial_\eta\big) \quad \text{and} \quad H_V = H_g \circ h_2,$$

where

$$H_g(\xi, \eta) = G(\xi^2 + \eta^2), \qquad G(z) = \frac{1}{2}\int_0^z g(s)\,ds.$$

Since the conjugacy h_2 is canonical and satisfies $H_V = H_g \circ h_2$, the energy levels $\{H_V = E\}$ and $\{H_g = E\}$ correspond exactly. Consequently, the period functions coincide when parametrized by the same energy value, and therefore

$$T_g(E) = T(E; X_g, H_g) = T(E; X_V, H_g \circ h_2) = T(E; X_V, H_V) = T_V(E) = T_f(E).$$

The period function of X_g is explicitly given by

$$T_g(E) = \frac{\pi}{G'(G^{-1}(E))} = \frac{d}{dE} G^{-1}(E).$$

Since $T_g(E) = T_f(E)$ and

$$T_f(E) = \frac{d}{dE} F^{-1}(E),$$

it follows that

$$\frac{d}{dE} G^{-1}(E) = \frac{d}{dE} F^{-1}(E).$$

As $G^{-1}(0) = F^{-1}(0) = 0$, we conclude that

$$G^{-1}(E) = \int_0^E \frac{d}{ds} G^{-1}(s)\,ds = \int_0^E \frac{d}{ds} F^{-1}(s)\,ds = F^{-1}(E),$$

for E sufficiently small. Hence $G^{-1} = F^{-1}$ near 0, and therefore $G = F$ and $g = f$ in a neighborhood of the origin. Consequently, the vector fields X_f and X_g coincide in a neighborhood of the origin.

Finally, define the local analytic diffeomorphism

$$h = h_2^{-1} \circ h_1.$$

Then

$$h_* X = (h_2^{-1} \circ h_1)_* X = (h_2^{-1})_* X_f = X_V.$$

Therefore, the original vector field X is locally analytically conjugate to the mechanical Hamiltonian system X_V, which completes the proof of Theorem 1.1. □

We note that, if the vector field X in Theorem 1.1 is Hamiltonian, then the analytic conjugacy constructed in the proof can be chosen to be canonical. Indeed, in this case the first normalization step may be carried out within the Hamiltonian category, while the second step of the construction is canonical by design. This observation leads to the following corollary.

Corollary 2.6 *Let H be a real-analytic function defined in a neighborhood of the origin and satisfying $H(0, 0) = 0$. Assume that the Hamiltonian vector field*

$$X_H = H_v \, \partial_u - H_u \, \partial_v$$

has a non-degenerate center at the origin. Then X_H is locally analytically conjugate, via a canonical (area-preserving) analytic diffeomorphism, to a mechanical Hamiltonian vector field of the form

$$X_V = y \, \partial_x - V'(x) \, \partial_y,$$

where V is an even real-analytic function defined in a neighborhood of the origin and satisfies

$$V(0) = V'(0) = 0, \qquad V''(0) > 0.$$

Proof The proof follows the same steps as the proof of Theorem 1.1, with the additional observation that, in the present Hamiltonian setting, the conjugacies can be chosen to be canonical.

Indeed, since X_H is a Hamiltonian vector field with a non-degenerate center at the origin, Lemma 2.5 ensures that the first conjugacy h_1, which brings X_H into rotational normal form, can be chosen canonical.

The second conjugacy h_2, which relates the rotational normal form to the mechanical Hamiltonian system X_V, is canonical by construction, since it is obtained within the Hamiltonian category.

Therefore, the composition

$$h = h_2^{-1} \circ h_1$$

is a canonical (area-preserving) analytic diffeomorphism, and conjugates X_H to the mechanical Hamiltonian vector field X_V. □

Remark 2.7 Alternatively, the above corollary also follows by applying the Fundamental Lemma 2.3 to the Hamiltonian vector field X_H and then invoking Corollary 4.3 of Mañosas–Villadelprat, since both systems have the same period function.

A linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$ is called a *linear involution* if $R \neq \mathrm{id}$ and $R^2 = \mathrm{id}$. An analytic vector field $X : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is said to be *reversible with respect to* R, or simply R*-reversible*, if

$$R \circ X(q) = -X \circ R(q) \qquad \text{for all } q \in \Omega.$$

Corollary 2.8 *Let $X = P\,\partial_u + Q\,\partial_v$ be a real-analytic vector field defined in a neighborhood of $(0,0)$, satisfying $X(0,0) = (0,0)$ and $\det\big(DX(0,0)\big) > 0$. Assume moreover that X is reversible with respect to the involution $R(u,v) = (u,-v)$. Then X is analytically conjugate, in a neighborhood of $(0,0)$, to the mechanical Hamiltonian vector field*

$$X_V = y\,\partial_x - V'(x)\,\partial_y,$$

where V is an even real-analytic function defined near the origin and satisfies

$$V(0) = V'(0) = 0, \qquad V''(0) > 0.$$

Proof Under these hypotheses, the origin is a non-degenerate center of X; see [10, Theorem 1]. The conclusion then follows directly from Theorem 1.1. □

The following example illustrates the constructive nature of Theorem 1.1 and shows explicitly how the period function determines the analytically conjugate mechanical system.

Example 2.9 Consider the vector field

$$X = G(u)\,v\,\partial_u \;-\; G(u)\,F'(u)\,\partial_v,$$

where

$$F(u) = \tfrac{1}{2}u^2, \qquad G(u) = 1 + u^2.$$

Since F has a strict non-degenerate minimum at $u = 0$ and $G(0) > 0$, the origin is a non-degenerate center of X. Moreover, the function

$$H(u,v) = \frac{1}{2}v^2 + F(u) = \frac{1}{2}v^2 + \frac{1}{2}u^2$$

is a first integral of X.

Let $E > 0$. The period of the periodic orbit of X contained in the energy level $\{H = E\}$ is given by

$$T(E) = \sqrt{2}\left(\int_{u_-(E)}^{0} \frac{du}{G(u)\sqrt{E - F(u)}} + \int_{0}^{u_+(E)} \frac{du}{G(u)\sqrt{E - F(u)}}\right),$$

where $u_-(E) < 0 < u_+(E)$ are the turning points determined by $F(u_\pm(E)) = E$.

Since F is even, the turning points satisfy $u_-(E) = -u_+(E)$. Moreover, since G is also even, the two integrals coincide, and the period simplifies to

$$T(E) = 2\sqrt{2}\int_0^{u_+(E)} \frac{du}{G(u)\sqrt{E - F(u)}}.$$

Setting $s = F(u)$, which defines an analytic change of variables on $[0, u_+(E)]$, we obtain

$$T(E) = 2\sqrt{2}\int_0^{E} \frac{(F^{-1})'(s)}{G(F^{-1}(s))\sqrt{E - s}}\, ds.$$

Since

$$F^{-1}(s) = \sqrt{2s}, \qquad (F^{-1})'(s) = \frac{\sqrt{2}}{2\sqrt{s}}, \qquad G(F^{-1}(s)) = 1 + 2s,$$

it follows that

$$T(E) = 2\int_0^{E} \frac{1}{(1+2s)\sqrt{s(E-s)}}\, ds = \frac{2\pi}{\sqrt{1+2E}}.$$

Now consider the potential

$$V(x) = \tfrac{1}{2}\tan^2 x.$$

Since V is even, the period function $T_V(E)$, parametrized by the energy levels $H_V = E$, is given by the analytic expression

$$T_V(E) = 2\sqrt{2}\int_0^{E} \frac{(V^{-1})'(s)}{\sqrt{E - s}}\, ds.$$

A direct computation shows that the period function associated with the vector field X_V satisfies

$$T_V(E) = \frac{2\pi}{\sqrt{1+2E}} = T(E).$$

Hence, Theorem 1 ensures that the vector fields X and X_V are analytically conjugate. Indeed, consider the analytic diffeomorphism

$$(x, y) = h(u, v) = \big(\arctan(u),\ v\big).$$

A straightforward computation shows that $X_V = h_* X$. Therefore, the vector fields X and X_V are analytically conjugate.

3 Conclusion

It is worth mentioning that the fact that two Hamiltonian systems with non-degenerate centers are analytically conjugate if and only if their period functions coincide is a

well-known result; see [9, Corollary 4.3]. However, this theorem holds only within the Hamiltonian setting and under canonical (area-preserving) transformations. In contrast, Theorem 1.1 applies to *any* analytic vector field with a non-degenerate center and relies only on general analytic diffeomorphisms. Thus our result extends the period-function classification far beyond the Hamiltonian category, showing that every analytic center can be locally realized as a one-degree-of–freedom mechanical system.

This work was motivated by the following question posed by C. Chicone (see [5]): under what conditions can a system of the form

$$u' = P(u, v), \qquad v' = Q(u, v),$$

with a non-degenerate center at $(0, 0)$ be transformed into a Hamiltonian system of the form

$$x' = y, \qquad y' = -V'(x)?$$

Our Theorem 1.1 provides a local analytic answer to this question.

Professor C. Chicone became aware of an earlier Portuguese version of this result (see [11]) and, in a private communication, encouraged the author to prepare and submit an English version for possible publication.

It is also worth noting that the present version differs substantially from the preliminary Portuguese version, where only the main theorem was presented. In this paper we include examples and additional material that clarify the constructive nature of the result and its relation with the period function.

A global counterpart to the question addressed here was later treated in the joint work of Grotta-Ragazzo and the author [4], which established global mechanical normal forms for planar vector fields possessing a non-degenerate center. That global result is conceptually related but methodologically distinct from the local analytic theory developed in the present paper.

Acknowledgements The author warmly thanks Professor C. Chicone for his interest in this work and for encouraging the preparation of an English version for publication.

Author Contributions F.J.S.N. is the sole author of this manuscript and was responsible for all aspects of the work, including conceptualization, formal analysis, writing, and preparation of examples.

Funding The Article Processing Charge (APC) for the publication of this research was funded by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) (ROR identifier: 00x0ma614). Not applicable.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflicts of Interest The author declares no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If

material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Poincaré, H.: Sur les courbes définies par les équations différentielles. Journal de mathématiques pures et appliquées **1**, 167–244 (1885)
2. Mardesic, P., Rousseau, C., Toni, B.: Linearization of isochronous centers. J. Differential Equations **121**(1), 67–108 (1995)
3. Chavarriga, J., Sabatini, M.: A survey of isochronous centers. Qualitative theory of Dynamical Systems **1**, 1–70 (1999)
4. Global normalizations for centers of planar vector fields: Grotta-Ragazzo, C., Nascimento, F.J.d.S. J. Differential Equations **415**, 701–721 (2025)
5. Chicone, C.: The monotonicity of the period function for planar hamiltonian vector fields. J. Differential Equations **69**(3), 310–321 (1987)
6. Alfawicka, B.: Inverse problem connected with half-period function analytic at the origin. Bull. Pol. Acad. Sci. Math. **32**(5–6), 267–274 (1984)
7. Treschev, D.V.: Hamiltonian systems with a functional parameter in the form of a potential. Russ. J. Math. Phys. **29**(3), 402–412 (2022)
8. Mañosas, F., Torres, P.J.: Two inverse problems for analytic potential systems. J. Differential Equations **245**(12), 3664–3673 (2008)
9. Manosas, F., Villadelprat, J.: Area-preserving normalizations for centers of planar hamiltonian systems. J. Differential Equations **179**(2), 625–646 (2002)
10. Teixeira, M.A., Yang, J.: The center-focus problem and reversibility. J. Differential Equations **174**(1), 237–251 (2001)
11. Dos Santos Nascimento, F.J.: Conjugação analítica entre sistemas diferenciais planares e sistemas potenciais. CQD - Revista Eletrônica Paulista de Matemática n/a, 24015 (2024)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.